ALGEBRA II Problems: Week 16 (group actions, orbits, stabilisers)

Epiphany Term 2014

Hwk: **Q1, 2** due Thursday, Mar 6, during the lectures.

1. The infinite dihedral group D_∞ is generated (as a subgroup of the group $S_{\mathbb{R}}$ of bijections : $\mathbb{R} \to \mathbb{R}$), by the translation $t(x) = x+1$ and the reflection $s(x) = -x$ of the real line. Work out its elements, and find the orbit and the stabilizer of each of the points 1, 1/2, 1/3.

2. Let H be a subgroup of a group G. Verify that the formula $(h, h')(x) = hxh'^{-1}$ defines an action of $H \times H$ on G. Find the orbit and the stabilizer of each element of G when $G = D_4$ and $H = \{ e, s \}$.

3. If G acts on X and on Y, show that the formula $g((x, y)) = (g(x), g(y))$ defines an action of G on $X \times Y$. Check that the stabilizer of (x, y) is the intersection of G_x and G_y. Give an example which shows this action need not be transitive even if G acts transitively on both X and Y. We call this action the *diagonal action* of G on $X \times Y$.
[A group action is said to be *transitive* if there is just one orbit.]

4. Let $X = \{1, 2, 3, 4\}$ and let G be the subgroup of S_4 generated by (1234) and (24). Work out the orbits and stabilizers for the diagonal action of G on $X \times X$.

5. Let n be *even*, and let D_n act on itself by conjugation. Find the orbits and stabilizers of the elements of D_n under this action.
In which sense does this case differ from the case when n is odd?

6. Let $V = \{e, (12)(34), (13)(24), (14)(23)\}$ act on A_4 (viewed as a subgroup of S_4, as usual) by conjugation.
Determine, for each $x \in A_4$, its orbit $V(x)$ and stabilizer V_x under this action and give a natural bijection between $V(x)$ and the set of cosets in V with respect to V_x.

7. Given an action of a group G on a set, show that every point of some orbit has the same stabilizer if and only if this stabilizer is a normal subgroup of G.

8. Show that the cosets of A_n in S_n are the set of even and the set of odd permutations. Deduce that if $n > 1$ then $|A_n| = \frac{1}{2}n!$.

9. Prove or give a counterexample to the following statement:
if a and b are two elements in a group G, then ab and ba have the same order.

10. Prove that if G is a finite group of *odd* order, then no $x \in G$, other than $x = e$, is conjugate to its inverse. Contrast this with the case of a dihedral group.

11.* Show that every group of order $4n + 2$ contains a subgroup of order $2n + 1$.
[Hint: Use Cayley's Theorem and Cauchy's Theorem and think odd and even.]