ALGEBRA II Problems: Week 15 (Cayley's Theorem, Group Actions)

Epiphany Term 2014

Hwk: **Q4, 8** (try **Q6** for extra points) due Thursday, Febr. 27 during the lectures.

1. Label the edges of a regular tetrahedron 1 to 6, so that each rotational symmetry of the tetrahedron produces an element of S_6. Work out the twelve elements of S_6 which occur in this way and check that they form a subgroup of S_6.

2. Which rotational symmetries of the cube
(a) have order 2?
(b) leave some principal diagonal pointwise fixed?
(c) leave some principal diagonal where it is but reverse its direction?
In each case describe the axis of the rotation and work out the cycle decomposition of the corresponding permutation of the four principal diagonals.

3. Carry out the procedure of the proof of Cayley's Theorem to obtain a subgroup of S_8 which is isomorphic to D_4.

4. Show that Cayley's Theorem, when applied to $\mathbb{R}$, produces the subgroup of $S_{\mathbb{R}}$ which consists of all translations of the real line.

5. Convert each element α of S_n into an element α_* of S_{n+2} as follows. The new permutation behaves just like the original on the integers $1, 2, ..., n$. If α is an even permutation then α_* fixes $n+1$ and $n+2$, whereas if α is odd α_* interchanges $n+1$ and $n+2$. Verify that α_* is always an *even* permutation and that the correspondence $\alpha \to \alpha_*$ defines an isomorphism from S_n to a subgroup of A_{n+2}. Work out this subgroup when $n = 3$.
Upshot: If G is a finite group of order n, prove that G is isomorphic to a subgroup of the *alternating group* A_{n+2}.

6. Draw five dodecahedra, with a different inscribed cube in each one. Use your pictures to examine how these cubes are permuted by the following rotations:
(a) A rotation of the dodecahedron through $2\pi/5$ about an axis which joins the centres of a pair of opposite faces.
(b) A rotation through π about an axis determined by the mid points of a pair of opposite edges.
(c) A rotation through $2\pi/3$ about an axis which joins a pair of opposite vertices.
How many of these types of rotations are there?
Convince yourself that the rotational symmetry group of the dodecahedron is isomorphic to A_5.

7. Explain which (if any) of the following describe group actions of G on X.
(a) $G = \mathbb{Z}$, $X = \mathbb{R}$, for $n \in \mathbb{Z}$ and $s \in \mathbb{R}$, put $n(s) = ns$.
(b) $G = \mathrm{GL}_2(\mathbb{R})$, $X = \mathrm{Mat}_2(\mathbb{R})$, for $A \in G$ and $M \in X$, put $A(M) = (1/2)(AM + MA)$.
(c) G and X as in (b), this time for $A \in G$ and $M \in X$, put $A(M) = MA$.
(d) $G = \mathbb{R}^*$, $X = \mathbb{R}^2$, for $r \in G$ and $(a, b) \in X$, put $r(a, b) = (ra, 0)$.

8. Let G be the subgroup of S_8 generated by $(123)(45)$ and (78). Then G acts as a group of permutations of the set $X = \{1, 2, ..., 8\}$. Work out the elements of G, and calculate the orbit and the stabilizer of every integer in X.