ALGEBRA II Problems: Week 14 (Identifying, distinguishing groups)

Epiphany Term 2014

Hwk: **Q4, 6, 11** due Thursday Feb 20, after the lectures.

1. Show that a k-cycle in S_n $(1 \leq k \leq n)$ has order k.
Moreover, show that the product of a k_1-cycle and a k_2-cycle which are disjoint has as its order the least common multiple $\mathrm{lcm}(k_1, k_2)$.

2. For any $\sigma \in S_n$ $(n \geq 2)$ put

$$P_{n,\sigma} = \prod_{1 \leq i < j \leq n} \left(x_{\sigma(i)} - x_{\sigma(j)}\right),$$

where $x_1, \ldots, x_n$ are independent variables.
Show that, for any transposition $\tau \in S_n$, one has

$$P_{n,\tau\sigma} = -P_{n,\sigma}\,.$$

[Hint: Try to prove it for $\sigma = e$ first, and investigate which ones of the factors $(x_i - x_j)$ change their sign under τ. Do you see a pattern?]

3. Check that none of the following groups are isomorphic to one another:

$$\mathbb{Z}_8, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad D_4, \quad A_4\,.$$

4. Which of the following groups are isomorphic to one another? Justify your answers.
(a) $\mathbb{Z}_4 \times \mathbb{Z}_3, \quad \mathbb{Z}_6 \times \mathbb{Z}_2, \quad V \times \mathbb{Z}_3, \quad S_3 \times \mathbb{Z}_2.$
(b) $\mathbb{Z}_6 \times \mathbb{Z}_4, \quad D_4 \times \mathbb{Z}_3, \quad D_{12}, \quad A_4 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times D_6, \quad S_4, \quad \mathbb{Z}_{12} \times \mathbb{Z}_2.$
(Here V, as usual, denotes the Klein 4-group.)

5. Show that, for a prime p, the direct product of $\mathbb{Z}_p$ with itself is not isomorphic to $\mathbb{Z}_{p^2}$.

6. Show that A_4 has no subgroup of order 6.
Why does this show that the converse of Lagrange's Theorem is false?

7. Show that the integers 1,3,7,9,11,13,17,19 form a group under multiplication mod 20. Identify this group as a direct product of cyclic groups.

8. Produce a specific isomorphism between S_3 and D_3. How many different isomorphisms are there from S_3 to D_3?

9. Let G and H be groups. Show that $G \times H$ is abelian if and only if both G and H are abelian. If $G \times H$ is cyclic, prove that G and H are both cyclic.

10. Show that $\mathbb{Z} \times \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$.

11. Prove that a group G is abelian if and only if the correspondence $x \mapsto x^{-1}$ is an isomorphism from G to G.

12. If G is a group and if g is an element of G, show that the function $\phi : G \to G$ defined by $\phi(x) = gxg^{-1}$ is an isomorphism. Work out this isomorphism when G is A_4 and g is the permutation (123).