ALGEBRA II Problems: Week 13
(Orders, homomorphisms, cosets, cycles)

Epiphany Term 2014

Hwk: **Q1, Q7**, due Thu Feb 13. Tutorials: **Q2 (2nd, 3rd), Q3, Q5, Q6**

1. (a) Show that a k-cycle can be written as a product of transpositions:
$$(i_1\, i_2\, \ldots\, i_k) = (i_1\, i_k)(i_1\, i_{k-1}) \ldots (i_1\, i_2) \qquad (k \geq 2)\,.$$
For $k > 2$, find a different such product of transpositions.

   (b) Using (a), or otherwise, find the inverse of the cycle $(i_1\, i_2\, \ldots\, i_k)$.

2. Express each of the following three permutations as (i) a product of disjoint cycles and (ii) a product of transpositions:
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 4 & 1 & 8 & 2 & 3 & 5 \end{pmatrix}; \quad (4\,5\,6\,8)(1\,2\,4\,5); \quad (6\,2\,4)(2\,5\,3)(8\,7\,6)(4\,5).$$

3. Let H be a subgroup of G. Show that gHg^{-1} is also a subgroup of G for any $g \in G$. Then show that every left coset of H is equal to a right coset of *some* subgroup (not necessarily H) of G.

4. How many different 5-cycles are there in S_5? [Justify your answer.]

5. Consider the subset $W = \{\, e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\,\}$ of S_4.

   (a) Show that W forms a subgroup of S_4.

   (b) Is W isomorphic to $\mathbf{Z_4}$ or to $\mathbf{Z_2} \times \mathbf{Z_2}$? [Justify your answer.]

   (c) Show that W is isomorphic to the group of plane symmetries of a chess board.
   [Hint: label four distinguished points on the board by 1,. . . ,4, resp.]

6. Find the centre of S_n for $n \geq 3$.

7. (a) Show that the order of each element g of a group G divides the order of G.

   (b)* Show that there can only be two types of group of order 6, up to isomorphism. [Hint: one of them is abelian, the other one is not.]
   (Possible tools: What are the possible orders of elements—how many can there be each? How do possible normal subgroups look like? A multiplication table has to list each element in each row and column. You might even want to use the Chinese Remainder Theorem to identify two candidates.)

8. Find a subgroup of S_4 which contains six elements. How many subgroups of order six are there in S_4? (You may use that a group of order six is isomorphic to either $\mathbf{Z_6}$ or S_3. What are the orders of elements in each?)

9. For each of the groups $\mathbf{Z_6}$, S_3, D_4, $\mathbf{Z_2} \times \mathbf{Z_2}$ either find a subgroup of D_6 that is isomorphic to it, together with a specific isomorphism between the two, or explain why no such subgroup exists.