PERFECT FORMS, K-THEORY AND THE COHOMOLOGY OF MODULAR GROUPS

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Contents

Abstract. For $N = 5, 6$ and 7, using the classification of perfect quadratic forms, we compute the homology of the Voronoï cell complexes attached to the modular groups $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$. From this we deduce the rational cohomology of those groups and some information about $K_m(\mathbb{Z})$, when $m = 5, 6$ and 7.

1. Introduction

Let $N \geqslant 1$ be an integer and let $SL_N(\mathbb{Z})$ be the modular group of integral matrices with determinant one. Our goal is to compute its cohomology groups with trivial coefficients, i.e. $H^q(SL_N(\mathbb{Z}), \mathbb{Z})$. The case $N = 2$ is well-known and follows from the fact that $SL_2(\mathbb{Z})$ is the amalgamated product of two finite cyclic groups ([29], [7], II.7, Ex.3, p.52). The case $N = 3$ was done in [31]: for any $q > 0$ the group $H^q(SL_3(\mathbb{Z}), \mathbb{Z})$ is killed by 12. The case $N = 4$ has been studied by Lee and Szczarba in [19]: modulo 2, 3 and 5–torsion, the cohomology group $H^q(SL_4(\mathbb{Z}), \mathbb{Z})$ is trivial whenever $q > 0$, except that $H^3(SL_4(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}$. In Theorem 7.3 below, we solve the cases $N = 5, 6$ and 7.

For these calculations we follow the method of [19], i.e. we use the perfect forms of Voronoï. Recall from [34] and [20] that a perfect form in N variables is a positive definite real quadratic form h on $\mathbb{R}^N$ which is uniquely determined (up to a scalar) by its set of integral minimal vectors. Voronoï proved in [34] that there are finitely many perfect forms of rank N, modulo the action of $SL_N(\mathbb{Z})$. These are known for $N \leqslant 8$ (see §2 below).

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Voronoï used perfect forms to define a cell decomposition of the space X_N^* of positive real quadratic forms, the kernel of which is defined over $\mathbb{Q}$. This cell decomposition (cf. §3) is invariant under $SL_N(\mathbb{Z})$, hence it can be used to compute the equivariant homology of X_N^* modulo its boundary. On the other hand, this equivariant homology turns out to be isomorphic to the groups $H_q(SL_N(\mathbb{Z}), St_N)$, where St_N is the Steinberg module (see [6] and §3.4 below). Finally, Borel–Serre duality [6] asserts that the homology $H_*(SL_N(\mathbb{Z}), St_N)$ is dual to the cohomology $H^*(SL_N(\mathbb{Z}), \mathbb{Z})$ (modulo torsion).

To perform these computations for $N \leqslant 7$, we needed the help of a computer. The reason is that the Voronoï cell decomposition of X_N^* gets soon very complicated when N increases. For instance, when $N = 7$, there are more than two million orbits of cells of dimension 18, modulo the action of $SL_N(\mathbb{Z})$ (see Figure 2 below). For this purpose, we have developed a C library [23], which uses PARI [22] for some functionalities. The algorithms are based on exact methods. As a result we get the full Voronoï cell decomposition of the spaces X_N^* for $N \leqslant 7$ (with either $GL_N(\mathbb{Z})$ or $SL_N(\mathbb{Z})$ action). Those decompositions are summarized in the figures and tables below. The computations were done on several computers using different processor architectures (which is useful for checking the results) and for $N = 7$ the overall computational time was more than a year.

The paper is organized as follows. In §2, we recall the Voronoï theory of perfect forms. In §3, we introduce a complex of abelian groups that we call the "Voronoï complex" which computes the homology groups $H_q(SL_N(\mathbb{Z}), St_N)$. In §4, we explain how to get an explicit description of the Voronoï complex in rank $N = 5$, 6 or 7, starting from the description of perfect forms available in the literature (especially in the work of Jaquet [15]). In Figures 1 and 2 we display the rank of the groups in the Voronoï complex and in Tables 1–5 we give the elementary divisors of its differentials. The homology of the Voronoï complex (hence the groups $H_q(SL_N(\mathbb{Z}), St_N)$) follows from this. It is given in Theorem 4.3.

We found two methods to test whether our computations are correct. First, checking that the virtual Euler characteristic of $SL_N(\mathbb{Z})$ vanishes leads to a mass formula for the orders of the stabilizers of the cells of X_N^* (cf. §4.5). Second, the identity $d_{n-1} \circ d_n = 0$ for the differentials in the Voronoï complex is a non-trivial equality when these differentials are written as explicit (large) matrices.

In §5 we give an explicit formula for the top homology group of the Voronoï complex (Theorem 5.1). In §6 we prove that the Voronoï complex of $GL_5(\mathbb{Z})$ is a direct factor of the Voronoï complex of $GL_6(\mathbb{Z})$ shifted by one. In §7 we explain how to compute the cohomology of $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$ (modulo torsion) from our results on the homology of the Voronoï complex in §4. Our main result is stated in Theorem 7.3. In §8 we compute some homology groups of $GL_N(\mathbb{Z})$ with coefficients the Steinberg module. In §9, we use these results to get some information on $K_m(\mathbb{Z})$, when $m = 5, 6$ and 7. Some of these results had already been announced in [10].

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Notation: For any positive integer n we let $\mathcal{S}_n$ be the class of finite abelian groups the order of which has only prime factors less than or equal to n.

2. VORONOÏ'S REDUCTION THEORY

2.1. **Perfect forms.** Let $N \geqslant 2$ be an integer. We let C_N be the set of positive definite real quadratic forms in N variables. Given $h \in C_N$, let $m(h)$ be the finite set of minimal vectors of h, i.e. vectors $v \in \mathbb{Z}^N$, $v \neq 0$, such that $h(v)$ is minimal. A form h is called *perfect* when $m(h)$ determines h up to scalar: if $h' \in C_N$ is such that $m(h') = m(h)$, then h' is proportional to h.

***Example* 2.1.** The form $h(x, y) = x^2 + y^2$ has minimum 1 and precisely 4 minimal vectors $\pm(1, 0)$ and $\pm(0, 1)$. This form is not perfect, because there is an infinite number of positive definite quadratic forms having these minimal vectors, namely the forms $h(x, y) = x^2 + axy + y^2$ where a is a non-negative real number less than 1. By contrast, the form $h(x, y) = x^2 + xy + y^2$ has also minimum 1 and has exactly 6 minimal vectors, viz. the ones above and $\pm(1, -1)$. This form is perfect, the associated lattice is the "honeycomb lattice".

Denote by C_N^* the set of non-negative real quadratic forms on $\mathbb{R}^N$ the kernel of which is spanned by a proper linear subspace of $\mathbb{Q}^N$, by X_N^* the quotient of C_N^* by positive real homotheties, and by $\pi : C_N^* \to X_N^*$ the projection. Let $X_N = \pi(C_N)$ and $\partial X_N^* = X_N^* - X_N$. Let Γ be either $GL_N(\mathbb{Z})$ or $SL_N(\mathbb{Z})$. The group Γ acts on C_N^* and X_N^* on the right by the formula

$$h \cdot \gamma = \gamma^t \, h \, \gamma \, , \quad \gamma \in \Gamma \, , \; h \in C_N^* \, ,$$

where h is viewed as a symmetric matrix and γ^t is the transpose of the matrix γ. Voronoï proved that there are only finitely many perfect forms modulo the action of Γ and multiplication by positive real numbers ([34], Thm. p.110).
The following table gives the current state of the art on the enumeration of perfect forms.

rank	1	2	3	4	5	6	7	8	9
#classes	1	1	1	2	3	7	33	10916	$\geqslant$ 500000

The classification of perfect forms of rank 8 was achieved by Dutour, Schürmann and Vallentin in 2005 [9], [28]. They have also shown that in rank 9 there are at least 500000 classes of perfect forms. The corresponding classification for rank 7 was completed by Jaquet in 1991 [15], for rank 6 by Barnes [2], for rank 5 and 4 by Korkine and Zolotarev [16], [17], for dimension 3 by Gauss [13] and for dimension 2 by Lagrange [18]. We refer to the book of Martinet [20] for more details on the results up to rank 7.

2.2. **A cell complex.** Given $v \in \mathbb{Z}^N - \{0\}$ we let $\hat{v} \in C_N^*$ be the form defined by

$$\hat{v}(x) = (v \mid x)^2 \, , \; x \in \mathbb{R}^N \, ,$$

where $(v \mid x)$ is the scalar product of v and x. The *convex hull in* X_N^* of a finite subset $B \subset \mathbb{Z}^N - \{\mathbf{0}\}$ is the subset of X_N^* which is the image under π of the quadratic forms $\sum_j \lambda_j \, \hat{v}_j \in C_N^*$, where $v_j \in B$ and $\lambda_j \geqslant 0$. For any perfect form h, we let $\sigma(h) \subset X_N^*$ be the convex hull of the set $m(h)$ of its minimal vectors. Voronoï proved in [34], §§8-15, that the cells $\sigma(h)$ and their intersections, as h runs over all perfect forms, define a cell decomposition of X_N^*, which is invariant under the action of Γ. We endow X_N^* with the corresponding CW-topology. If τ is a closed cell in X_N^* and h a perfect form with $\tau \subset \sigma(h)$, we let $m(\tau)$ be the set of vectors v in

$m(h)$ such that $\hat{v}$ lies in τ. Any closed cell τ is the convex hull of $m(\tau)$, and for any two closed c ells τ, τ' in X_N^* we have $m(\tau) \cap m(\tau') = m(\tau \cap \tau')$.

3. THE VORONOÏ COMPLEX

3.1. **An explicit differential for the Voronoi complex.** Let $d(N) = N(N+1)/2-1$ be the dimension of X_N^* and $n \leqslant d(N)$ a natural integer. We denote by $\Sigma_n^\star = \Sigma_n^\star(\Gamma)$ a set of representatives, modulo the action of Γ, of those cells of dimension n in X_N^* which meet X_N, and by $\Sigma_n = \Sigma_n(\Gamma) \subset \Sigma_n^\star(\Gamma)$ the cells σ for which any element of the stabilizer Γ_σ of σ in Γ preserves the orientation. Let V_n be the free abelian group generated by Σ_n. We define as follows a map

$$d_n : V_n \to V_{n-1} .$$

For each closed cell σ in X_N^* we fix an orientation of σ, i.e. an orientation of the real vector space $\mathbb{R}(\sigma)$ of symmetric matrices spanned by the forms $\hat{v}$ with $v \in m(\sigma)$. Let $\sigma \in \Sigma_n$ and let τ' be a face of σ which is equivalent under Γ to an element in Σ_{n-1} (i.e. τ' neither lies on the boundary nor has elements in its stabilizer reversing the orientation). Given a positive basis B' of $\mathbb{R}(\tau')$ we get a basis B of $\mathbb{R}(\sigma) \supset \mathbb{R}(\tau')$ by appending to B' a vector $\hat{v}$, where $v \in m(\sigma) - m(\tau')$. We let $\varepsilon(\tau', \sigma) = \pm 1$ be the sign of the orientation of B in the oriented vector space $\mathbb{R}(\sigma)$ (this sign does not depend on the choice of v).

Next, let $\tau \in \Sigma_{n-1}$ be the (unique) cell equivalent to τ' and let $\gamma \in \Gamma$ be such that $\tau' = \tau \cdot \gamma$. We define $\eta(\tau, \tau') = 1$ (resp. $\eta(\tau, \tau') = -1$) when γ is compatible (resp. incompatible) with the chosen orientations of $\mathbb{R}(\tau)$ and $\mathbb{R}(\tau')$.

Finally we define

$$d_n(\sigma) = \sum_{\tau \in \Sigma_{n-1}} \sum_{\tau'} \eta(\tau, \tau')\, \varepsilon(\tau', \sigma)\, \tau \,, \tag{1}$$

where τ' runs through the set of faces of σ which are equivalent to τ.

3.2. **A spectral sequence.** According to [7], VII.7, there is a spectral sequence E_{pq}^r converging to the equivariant homology groups $H_{p+q}^\Gamma(X_N^*, \partial X_N^*; \mathbb{Z})$ of the homology pair $(X_N^*, \partial X_N^*)$, and such that

$$E_{pq}^1 = \bigoplus_{\sigma \in \Sigma_p^\star} H_q(\Gamma_\sigma, \mathbb{Z}_\sigma) \,,$$

where $\mathbb{Z}_\sigma$ is the orientation module of the cell σ and, as above, $\Sigma_p^\star$ is a set of representatives, modulo Γ, of the p-cells σ in X_N^* which meet X_N. Notice that the action of Γ_σ on $\mathbb{Z}_\sigma$ is given by η described above. Since σ meets X_N, its stabilizer Γ_σ is finite and, by Lemma 7.1 in §7 below, the order of Γ_σ is divisible only by primes $p \leqslant N+1$. Therefore, when q is positive, the group $H_q(\Gamma_\sigma, \mathbb{Z}_\sigma)$ lies in $\mathcal{S}_{N+1}$.

When Γ_σ happens to contain an element which changes the orientation of σ, the group $H_0(\Gamma_\sigma, \mathbb{Z}_\sigma)$ is killed by 2, otherwise $H_0(\Gamma_\sigma, \mathbb{Z}_\sigma) \cong \mathbb{Z}_\sigma)$. Therefore, modulo $\mathcal{S}_2$, we have

$$E_{n\,0}^1 = \bigoplus_{\sigma \in \Sigma_n} \mathbb{Z}_\sigma \,,$$

and the choice of an orientation for each cell σ gives an isomorphism between $E_{n\,0}^1$ and V_n.

3.3. **Comparison.** We claim that the differential

$$d_n^1 : E_{n\,0}^1 \to E_{n-1,0}^1$$

coincides, up to sign, with the map d_n defined in 3.1. According to [7], VII, Prop. (8.1), the differential d_n^1 can be described as follows.

Let $\sigma \in \Sigma_n^\star$ and let τ' be a face of σ. Consider the group $\Gamma_{\sigma\tau'} = \Gamma_\sigma \cap \Gamma_{\tau'}$ and denote by

$$t_{\sigma\tau'} : H_*(\Gamma_\sigma, \mathbb{Z}_\sigma) \to H_*(\Gamma_{\sigma\tau'}, \mathbb{Z}_\sigma)$$

the transfer map. Next, let

$$u_{\sigma\tau'} : H_*(\Gamma_{\sigma\tau'}, \mathbb{Z}_\sigma) \to H_*(\Gamma_{\tau'}, \mathbb{Z}_{\tau'})$$

be the map induced by the natural map $\mathbb{Z}_\sigma \to \mathbb{Z}_{\tau'}$, together with the inclusion $\Gamma_{\sigma\tau'} \subset \Gamma_{\tau'}$. Finally, let $\tau \in \Sigma_{n-1}^\star$ be the representative of the Γ-orbit of τ', let $\gamma \in \Gamma$ be such that $\tau' = \tau \cdot \gamma$, and let

$$v_{\tau'\tau} : H_*(\Gamma_{\tau'}, \mathbb{Z}_{\tau'}) \to H_*(\Gamma_\tau, \mathbb{Z}_\tau)$$

be the isomorphism induced by γ. Then the restriction of d_n^1 to $H_*(\Gamma_\sigma, \mathbb{Z}_\sigma)$ is equal, up to sign, to the sum

$$\sum_{\tau'} v_{\tau'\tau}\, u_{\sigma\tau'}\, t_{\sigma\tau'}\,, \tag{2}$$

where τ' runs over a set of representatives of faces of σ modulo Γ_σ.

To compare d_n^1 with d_n we first note that, when $\tau \in \Sigma_{n-1}$,

$$v_{\tau'\tau} : H_0(\Gamma_{\tau'}, \mathbb{Z}_{\tau'}) = \mathbb{Z} \to H_0(\Gamma_\tau, \mathbb{Z}_\tau) = \mathbb{Z}$$

is the multiplication by $\eta(\tau, \tau')$, as defined in §3.1. Next, when $\sigma \in \Sigma_n$, the map

$$u_{\sigma\tau'} : H_0(\Gamma_{\sigma\tau'}, \mathbb{Z}_\sigma) = \mathbb{Z}_\sigma = \mathbb{Z} \to H_0(\Gamma_{\tau'}, \mathbb{Z}_{\tau'}) = \mathbb{Z}$$

is the multiplication by $\varepsilon(\tau', \sigma)$, up to a sign depending on n only. Finally, the transfer map

$$t_{\sigma\tau'} : H_0(\Gamma_\sigma, \mathbb{Z}_\sigma) = \mathbb{Z} \to H_0(\Gamma_{\sigma\tau'}, \mathbb{Z}_\sigma) = \mathbb{Z}$$

is the multiplication by $[\Gamma_\sigma : \Gamma_{\sigma\tau'}]$. Multiplying the sum (2) by this number amounts to the same as taking the sum over all faces of σ as in (1). This proves that d_n coincides, up to sign, with d_n^1 on $E_{n\,0}^1 = V_n$. □

In particular, we get that $d_{n-1} \circ d_n = 0$. Note that this identity will give us a non-trivial test of our explicit computations of the complex.

Notation: The resulting complex $(V_\bullet, d_\bullet)$ will be denoted by Vor_Γ, and we call it the *Voronoï complex*.

3.4. **The Steinberg module.** Let T_N be the spherical Tits building of SL_N over $\mathbb{Q}$, i.e. the simplicial set defined by the ordered set of non-zero proper linear subspaces of $\mathbb{Q}^N$. The reduced homology $\tilde{H}_q(T_N, \mathbb{Z})$ of T_N with integral coefficients is zero except when $q = N - 2$, in which case

$$\tilde{H}_{N-2}(T_N, \mathbb{Z}) = \mathrm{St}_N$$

is by definition the Steinberg module [6]. According to [30], Prop. 1, the relative homology groups $H_q(X_N^*, \partial X_N^*; \mathbb{Z})$ are zero except when $q = N - 1$, and

$$H_{N-1}(X_N^*, \partial X_N^*; \mathbb{Z}) = \mathrm{St}_N\,.$$

From this it follows that, for all $m \in \mathbb{N}$,

$$H_m^\Gamma(X_N^*, \partial X_N^*; \mathbb{Z}) = H_{m-N+1}(\Gamma, \mathrm{St}_N)$$

(see e.g. [30], §3.1). Combining this equality with the previous sections we conclude that, modulo $\mathcal{S}_{N+1}$,

$$H_{m-N+1}(\Gamma, \mathrm{St}_N) = H_m(\mathrm{Vor}_\Gamma)\,. \tag{3}$$

4. The Voronoï complex in dimensions 5, 6 and 7

In this section, we explain how to compute the Voronoï complexes of rank $N \leqslant 7$.

4.1. **Checking the equivalence of cells.** As a preliminary step, we develop an effective method to check whether two cells σ and σ' of the same dimension are equivalent under the action of Γ. The cell σ (resp. σ') is described by its set of minimal vectors $m(\sigma)$ (resp. $m(\sigma')$). We let b (resp. b') be the sum of the forms $\hat{v}$ with $v \in m(\sigma)$ (resp. $m(\sigma')$). If σ and σ' are equivalent under the action of Γ the same is true for b and b', and the converse holds true since two cells of the same dimension are equal when they have an interior point in common.

To compare b and b' we first check whether or not they have the same determinant. In case they do, we let M (resp. M') be the set of numbers $b(x)$ with $x \in m(\sigma)$ (resp. $b'(x)$ with $x \in m(\sigma')$). If b and b' are equivalent, then the sets M and M' must be equal.

Finally, if $M = M'$ we check if b and b' are equivalent by applying an algorithm of Plesken and Souvignier [24] (based on an implementation of Souvignier).

4.2. **Finding generators of the Voronoï complex.** In order to compute Σ_n (and $\Sigma_n^\star$), we proceed as follows. Fix $N \leqslant 7$. Let $\mathcal{P}$ be a set of representatives of the perfect forms of rank N. A choice of $\mathcal{P}$ is provided by Jaquet [15]. Furthermore, for each $h \in \mathcal{P}$, Jaquet gives the list $m(h)$ of its minimal vectors, and the list of all perfect forms $h'\gamma$ (one for each orbit under $\Gamma_{\sigma(h)}$), where $h' \in \mathcal{P}$ and $\gamma \in \Gamma$, such that $\sigma(h)$ and $\sigma(h')\gamma$ share a face of codimension one. This provides a complete list C_h^1 of representatives of codimension one faces in $\sigma(h)$.

From this, one deduces the full list $\mathcal{F}_h^1$ of faces of codimension one in $\sigma(h)$ as follows: first list all the elements in the automorphism group $\Gamma_{\sigma(h)}$; this can be obtained by using a second procedure implemented by Souvignier [24] which gives generators for $\Gamma_{\sigma(h)}$. We represent the latter generators as elements in the symmetric group $\mathfrak{S}_M$, where M is the cardinality of $m(h)$, acting on the set $m(h)$ of minimal vectors. Using those generators, we let GAP [12] list all the elements of $\Gamma_{\sigma(h)}$, viewed as elements of the symmetric group above.

The next step is to create a shortlist $\mathcal{F}_h^2$ of codimension 2 facets of $\sigma(h)$ by intersecting all the translates under $\mathfrak{S}_M$ of codimension 1 facets with each member of C_h^1 and only keeping those intersections with the correct rank ($=d(N) - 2$). The resulting shortlist is reasonably small and we apply the procedure of 4.1 to reduce the shortlist to a set of representatives C_h^2 of codimension 2 facets.

We then proceed by induction on the codimension to define a list $\mathcal{F}_h^p$ of cells of codimension $p > 2$ in $\sigma(h)$. Given $\mathcal{F}_h^p$, we let $C_h^p \subset \mathcal{F}_h^p$ be a set of representatives for the action of Γ. We then let $\mathcal{F}_h^{p+1}$ be the set of cells $\varphi \cap \tau$, with $\varphi \in \mathcal{F}_h^1$, and $\tau \in C_h^p$. As a result, we get directly the cellular structure of the quotient space $(X_N^*, \partial X_N^*)/\Gamma$ without computing the full cellular structure of X_N^* which is not required (and of greater computational complexity).

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\Sigma_n^\star(GL_5(\mathbb{Z}))$	2	5	10	16	23	25	23	16	9	4	3						
$\Sigma_n\,(GL_5(\mathbb{Z}))$					1	7	6	1	0	2	3						
$\Sigma_n^\star(GL_6(\mathbb{Z}))$		3	10	28	71	162	329	589	874	1066	1039	775	425	181	57	18	7
$\Sigma_n\,(GL_6(\mathbb{Z}))$						3	46	163	340	544	636	469	200	49	5		
$\Sigma_n^\star(SL_6(\mathbb{Z}))$		3	10	28	71	163	347	691	1152	1532	1551	1134	585	222	62	18	7
$\Sigma_n\,(SL_6(\mathbb{Z}))$			3	10	18	43	169	460	815	1132	1270	970	434	114	27	14	7

FIGURE 1. Cardinality of Σ_n and $\Sigma_n^\star$ for $N = 5, 6$ (empty slots denote zero).

n	6	7	8	9	10	11	12	13	14	15	16
$\Sigma_n^\star$	6	28	115	467	1882	7375	26885	87400	244029	569568	1089356
Σ_n				1	60	1019	8899	47271	171375	460261	955128

n	17	18	19	20	21	22	23	24	25	26	27
$\Sigma_n^\star$	1683368	2075982	2017914	1523376	876385	374826	115411	24623	3518	352	33
Σ_n	1548650	1955309	1911130	1437547	822922	349443	105054	21074	2798	305	33

FIGURE 2. Cardinality of Σ_n and $\Sigma_n^\star$ for $GL_7(\mathbb{Z})$.

Next, we let $\Sigma_n^\star$ be a system of representatives modulo Γ in the union of the sets $C_h^{d(N)-n}, h \in \mathcal{P}$. We then compute generators of the stabilizer of each cell in $\Sigma_n^\star$ with the help of another algorithm developed by Plesken and Souvignier in [24], and we check whether all generators preserve the orientation of the cell. This gives us the set Σ_n as the set of those cells which pass that check.

Proposition 4.1. *The cardinality of Σ_n and $\Sigma_n^\star$ is displayed in Figure 1 for rank $N = 5, 6$ and in Figure 2 for rank $N = 7$.*

***Remark* 4.2.** The first line in Figure 1 has already been computed by Batut (cf. [3], p.409, second column of Table 2). The running time for the computation of the cell structure (with the differentials and the checking) for $N = 7$ using [23] was 18 months on several servers including quadri-processors computers, while for $N = 6$ this can be done in a few seconds.

4.3. **The differential.** The next step is to compute the differentials of the Voronoï complex by using formula (1) above. In Table 3, we give information on the differentials in the Voronoï complex of rank 6. For instance the second line, denoted d_{11}, is about the differential from V_{11} to V_{10}. In the bases Σ_{11} and Σ_{10}, this differential is given by a matrix A with $\Omega = 513$ non-zero entries, with $m = 46 = \text{card}(\Sigma_{10})$ rows and $n = 163 = \text{card}(\Sigma_{11})$ columns. The rank of A is 42, and the rank of its kernel is 121. The elementary divisors of A are 1 (multiplicity 40) and 2 (multiplicity 2).

A	Ω	n	m	rank	ker	elementary divisors
d_4	0	1	0	0	1	
d_5	1	1	1	1	0	1(1)
d_6	0	1	1	0	1	
d_7	0	0	1	0	0	
d_8	0	1	0	0	1	
d_9	2	2	1	1	1	2(1)

TABLE 1. Results on the rank and elementary divisors of the differentials for $SL_4(\mathbb{Z})$.

A	Ω	n	m	rank	ker	elementary divisors
d_8	0	1	0	0	1	
d_9	2	7	1	1	6	1(1)
d_{10}	18	6	7	5	1	1(4), 2(1)
d_{11}	5	1	6	1	0	1(1)
d_{12}	0	0	1	0	0	
d_{13}	0	2	0	0	2	
d_{14}	4	3	2	2	1	5(1), 15(1)

TABLE 2. Results on the rank and elementary divisors of the differentials for $GL_5(\mathbb{Z})$.

The cases of $SL_4(\mathbb{Z})$, $GL_5(\mathbb{Z})$ and $SL_6(\mathbb{Z})$ are treated in Table 1, Table 2 and Table 4, respectively.

Our results on the differentials in rank 7 are shown in Table 5. While the matrices are sparse, they are not sparse enough for efficient computation. They have a poor conditioning with some dense columns or rows (this is a consequence of the fact that the complex is not simplicial and non-simplicial cells can have a large number of non-trivial intersections with the faces). We have obtained full information on the rank of the differentials. For the computation of the elementary divisors complete results have been obtained in the case of matrices of d_n except for $n = 19$. For this case, the computational cost is currently too high. The computations have required a full year on a parallel computer (including checking). For $n = 19$ alone, the computational cost is equivalent to 3 CPU-years on a current processor. See [8, 33] for a detailed description of the computations.

4.4. **The homology of the Voronoï complexes.** From the computation of the differentials, we can determine the homology of Voronoï complex. Recall that if we have a complex of free abelian groups

$$\cdots \to \mathbb{Z}^\alpha \xrightarrow{f} \mathbb{Z}^\beta \xrightarrow{g} \mathbb{Z}^\gamma \to \cdots$$

with f and g represented by matrices, then the homology is

$$\ker(g)/\mathrm{Im}(f) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_\ell\mathbb{Z} \oplus \mathbb{Z}^{\beta-\mathrm{rank}(f)-\mathrm{rank}(g)},$$

where $d_1, \ldots, d_\ell$ are the elementary divisors of the matrix of f.

We deduce from Tables 1–5 the following result on the homology of the Voronoï complex.

A	Ω	n	m	rank	ker	elementary divisors
d_{10}	17	46	3	3	43	1(3)
d_{11}	513	163	46	42	121	1(40), 2(2)
d_{12}	2053	340	163	120	220	1(120)
d_{13}	4349	544	340	220	324	1(217), 2(3)
d_{14}	6153	636	544	324	312	1(320), 2(1), 6(2), 12(1)
d_{15}	5378	469	636	312	157	1(307), 2(3), 60(2)
d_{16}	2526	200	469	156	44	1(156)
d_{17}	597	49	200	44	5	1(41), 3(1), 6(1), 36(1)
d_{18}	43	5	49	5	0	1(5)

TABLE 3. Results on the rank and elementary divisors of the differentials for $GL_6(\mathbb{Z})$.

A	Ω	n	m	rank	ker	elementary divisors
d_7	12	10	3	3	7	1(3)
d_8	48	18	10	7	11	1(7)
d_9	140	43	18	11	32	1(11)
d_{10}	613	169	43	32	137	1(32)
d_{11}	2952	460	169	136	324	1(129), 2(6), 6(1)
d_{12}	7614	815	460	323	492	1(318), 2(3), 4(2)
d_{13}	12395	1132	815	491	641	1(491)
d_{14}	14966	1270	1132	641	629	1(637), 3(3), 12(1)
d_{15}	12714	970	1270	629	341	1(621), 2(5), 6(1), 60(2)
d_{16}	6491	434	970	339	95	1(338), 2(1)
d_{17}	1832	114	434	95	19	1(92), 3(2), 18(1)
d_{18}	257	27	114	19	8	1(17), 2(2)
d_{19}	62	14	27	8	6	1(7), 10(1)
d_{20}	28	7	14	6	1	1(1), 3(4), 504(1)

TABLE 4. Results on the rank and elementary divisors of the differentials for $SL_6(\mathbb{Z})$.

Theorem 4.3. *The non-trivial homology of the Voronoï complexes associated to $GL_N(\mathbb{Z})$ with $N = 5, 6$ modulo $\mathcal{S}_5$ is given by:*

$$H_n(\mathrm{Vor}_{GL_5(\mathbb{Z})}) \cong \mathbb{Z}, \quad \text{if } n = 9, 14\,,$$
$$H_n(\mathrm{Vor}_{GL_6(\mathbb{Z})}) \cong \mathbb{Z}, \quad \text{if } n = 10, 11, 15\,,$$

while in the case $SL_6(\mathbb{Z})$ we get, modulo $\mathcal{S}_7$, that

$$H_n(\mathrm{Vor}_{SL_6(\mathbb{Z})}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 10, 11, 12, 20\,, \\ \mathbb{Z}^2, & \text{if } n = 15\,. \end{cases}$$

Furthermore, for $N = 7$ we get, modulo $\mathcal{S}_7$, that

$$H_n(\mathrm{Vor}_{GL_7(\mathbb{Z})}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 12, 13, 18, 22, 27\,, \\ 0 & \text{otherwise}. \end{cases}$$

Notice that, if N is odd, $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$ have the same homology modulo $\mathcal{S}_2$. Notice also that, for simplicity, in the statement of the theorem we did not use the full information given by the list of elementary divisors in Tables 1–5.

A	Ω	n	m	rank	ker	elementary divisors
d_{10}	8	60	1	1	59	1
d_{11}	1513	1019	60	59	960	1 (59)
d_{12}	37519	8899	1019	960	7939	1 (958), 2 (2)
d_{13}	356232	47271	8899	7938	39333	1 (7937), 2 (1)
d_{14}	1831183	171375	47271	39332	132043	1 (39300), 2 (29), 4 (3)
d_{15}	6080381	460261	171375	132043	328218	1(131993), 2(46), 12 (4)
d_{16}	14488881	955128	460261	328218	626910	1 (328183), 2 (33), 4(1), 12(1)
d_{17}	25978098	1548650	955128	626910	921740	1 (626857), 2(52), 4 (1)
d_{18}	35590540	1955309	1548650	921740	1033569	1 (921637), 2 (100), 42 (2), 252 (1)
d_{19}	37322725	1911130	1955309	1033568	877562	1 (1033458), 2 (110)
d_{20}	29893084	1437547	1911130	877562	559985	1 (877526), 2 (33), 6 (3)
d_{21}	18174775	822922	1437547	559985	262937	1 (559895), 2 (88), 6 (2)
d_{22}	8251000	349443	822922	262937	86506	1 (262835), 2 (98), 4 (3), 12 (1)
d_{23}	2695430	105054	349443	86505	18549	1 (86488), 2 (12), 6 (3), 42 (1), 84 (1)
d_{24}	593892	21074	105054	18549	2525	1 (18544), 2 (4), 4 (1)
d_{25}	81671	2798	21074	2525	273	1 (2507), 2 (18)
d_{26}	7412	305	2798	273	32	1 (258), 2 (7), 6 (7), 36 (1)
d_{27}	600	33	305	32	1	1 (23), 2 (4), 28 (3), 168 (1), 2016 (1)

TABLE 5. Results on the rank and elementary divisors of the differentials for $GL_7(\mathbb{Z})$, middle entries are cited from the thesis of A. Urbanska [33]. The elementary divisors for d_{19} were computed by B. Boyer and J.-G. Dumas using refinements of the techniques described in [8].

4.5. **Mass formulae for the Voronoï complex.** Let $\chi(SL_N(\mathbb{Z}))$ be the virtual Euler characteristic of the group $SL_N(\mathbb{Z})$. It can be computed in two ways. First, the mass formula in [7] gives

$$\chi(SL_N(\mathbb{Z})) = \sum_{\sigma\in E}(-1)^{\dim(\sigma)}\frac{1}{|\Gamma_\sigma|} = \sum_{n=N}^{d(N)}(-1)^n \sum_{\sigma\in\Sigma_n^\star}\frac{1}{|\Gamma_\sigma|}\,,$$

where E is a family of representatives of the cells of the Voronoï complex of rank N modulo the action of $SL_N(\mathbb{Z})$, and Γ_σ is the stabilizer of σ in $SL_N(\mathbb{Z})$. Second, by a result of Harder [14], we know that

$$\chi(SL_N(\mathbb{Z})) = \prod_{k=2}^{N}\zeta(1-k)\,,$$

hence $\chi(SL_N(\mathbb{Z})) = 0$ if $N \geqslant 3$.

A non-trivial check of our computations is to test the compatibility of these two formulas, and the corresponding check for rank $N = 5$ had been performed by Batut (cf. [3], where a proof of an analogous statement, for any N, but instead pertaining to *well-rounded* forms, which in our case are precisely the ones in $\Sigma_\bullet^\star$, is attributed to Bavard [4]).

If we add together the terms $\frac{1}{|\Gamma_\sigma|}$ for cells σ of the same dimension to a single term, then we get for $N = 6$, starting with the top dimension,

$$
\begin{aligned}
&\frac{45047}{1451520} - \frac{10633}{11520} + \frac{6425}{576} - \frac{12541}{192} \\
&+\frac{7438673}{34560} - \frac{3841271}{8640} + \frac{9238}{15} - \frac{266865}{448} + \frac{14205227}{34560} - \frac{14081573}{69120} \\
&+\frac{830183}{11520} - \frac{205189}{11520} + \frac{61213}{20736} - \frac{1169}{3840} + \frac{17}{1008} - \frac{1}{2880} \\
&= \chi(SL_6(\mathbb{Z})) = 0\,.
\end{aligned}
$$

For $N = 7$ we obtain similarly

$$
\begin{aligned}
&-\frac{290879}{107520} + \frac{13994381}{103680} - \frac{31815503}{13824} + \frac{1362329683}{69120} - \frac{6986939119}{69120} \\
&+\frac{7902421301}{23040} - \frac{340039739981}{414720} + \frac{174175928729}{120960} - \frac{132108094091}{69120} \\
&+\frac{27016703389}{13824} - \frac{13463035571}{8640} + \frac{14977461287}{15360} - \frac{22103821919}{46080} \\
&+\frac{8522164169}{46080} - \frac{17886026827}{322560} + \frac{1764066533}{138240} - \frac{101908213}{46080} + \frac{12961451}{46080} \\
&-\frac{10538393}{414720} + \frac{162617}{103680} - \frac{721}{11520} + \frac{43}{32256} \\
&= \chi\,(SL_7(\mathbb{Z})) = 0\,.
\end{aligned}
$$

5. EXPLICIT HOMOLOGY CLASSES

5.1. **Equivariant fundamental classes.**

Theorem 5.1. *The top homology group $H_{d(N)}(\mathrm{Vor}_{SL_N(\mathbb{Z})} \otimes \mathbb{Q})$ has dimension 1. When $N = 4, 5, 6$ or 7, it is represented by the cycle*

$$\sum_{\sigma} \frac{1}{|\Gamma_\sigma|}[\sigma]\,,$$

where σ runs through the perfect forms of rank N and the orientation of each cell is inherited from the one of X_N/Γ.

Proof. The first assertion is clear since, by (3) above and (6) below we have

$$H_{d(N)}(\mathrm{Vor}_{SL_N(\mathbb{Z})} \otimes \mathbb{Q}) \cong H_{d(N)-N+1}(SL_N(\mathbb{Z}), St_N \otimes \mathbb{Q}) \cong H^0(SL_N(\mathbb{Z}), \mathbb{Q}) \cong \mathbb{Q}\,.$$

In order to prove the second claim, write the differential between codimension 0 and codimension 1 cells as a matrix A of size $n_1 \times n_0$, with $n_i = |\Sigma_{d(N)-i}(\Gamma)|$ denoting the number of codimension i cells in the Voronoï cell complex. It can be checked that in each of the n_1 rows of A there are precisely two non-zero entries. Moreover, the absolute value of the (i, j)-th entry of A is equal to the quotient $|\Gamma_{\sigma_j}|/|\Gamma_{\tau_i}|$ (an *integer*), where $\sigma_j \in \Sigma_{d(N)}(\Gamma)$ and $\tau_i \in \Sigma_{d(N)-1}(\Gamma)$. Finally, one can multiply some columns by -1 (which amounts to changing the orientation of the corresponding codimension 0 cell) in such a way that each row has exactly one positive and one negative entry. □

***Example* 5.2.** For $N = 5$ the differential matrix d_{14} (cf. Table 2) between codimension 0 and codimension 1 is given by

$$\begin{pmatrix} 40 & 0 & -15 \\ 40 & -15 & 0 \end{pmatrix},$$

so the kernel is generated by $(3, 8, 8) = 11520\,(\frac{1}{3840}, \frac{1}{1440}, \frac{1}{1440})$, while the orders of the three automorphism groups are 3840, 1440 and 1440, respectively.

***Example* 5.3.** Similarly, the differential $d_{20} : V_{20} \to V_{19}$ for rank $N = 6$ (cf. Table 3) is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 96 & 0 & 0 & 0 & -21 \\ 3240 & 0 & 0 & 0 & -21 & 0 & 0 \\ 0 & 0 & 1440 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 18 & 0 & -6 & 0 \\ -12960 & 0 & 0 & 0 & 0 & 12 & 0 \\ -3240 & 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & -360 & 0 & 1 & 0 & 0 & 0 \\ -4320 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 960 & -6 & 0 & 0 & 0 \\ 0 & -216 & 96 & 0 & 0 & 0 & 0 \\ -45 & 45 & 0 & 0 & 0 & 0 & 0 \\ -2592 & 0 & 1152 & 0 & 0 & 0 & 0 \\ -3240 & 0 & 1440 & 0 & 0 & 0 & 0 \\ -432 & 0 & 192 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its kernel is generated by

$$(28, 28, 63, 10080, 4320, 30240, 288)$$

while the orders of the corresponding automorphism groups are, respectively,

$$103680, 103680, 46080, 288, 672, 96, 10080\,,$$

and we note that $28 \cdot 103680 = 63 \cdot 46080 = 10080 \cdot 288 = 4320 \cdot 672 = 30240 \cdot 96$.

5.2. **An explicit non-trivial homology class for rank $N = 5$.** The integer kernel of the 7×1-matrix of d_9 for $GL_5(\mathbb{Z})$, given by $(0, 0, 0, 0, -1, 0, 1)$, is spanned by the image of d_{10} (the latter being given, up to permutation of rows and columns, by the transpose of the matrix (4) below), together with $(2, 1, -1, -1, -1, 1, 1)$. The latter vector therefore provides the coefficients of a non-trivial homology class in $H_9(\mathrm{Vor}_{GL_5(\mathbb{Z})}) \cong H^5(GL_5(\mathbb{Z}), \mathbb{Z})$ (modulo $\mathcal{S}_5$), given as a linear combination of cells (in terms of minimal vectors) as follows:

$$\begin{aligned}
&2\,\varphi([e_1, e_2, \overline{e}_{23}, \overline{e}_{13}, e_3, \overline{e}_{34}, \overline{e}_{14}, \overline{e}_{45}, \overline{e}_{35}, \overline{e}_{25}]) \\
&+ \varphi([e_1, e_2, e_3, e_4, e_{24}, e_{34}, e_5, e_{15}, e_{35}, e_{1245}]) \\
&- \varphi([e_1, \overline{e}_{12}, e_2, \overline{e}_{23}, e_3, \overline{e}_{34}, \overline{e}_{14}, \overline{e}_{45}, \overline{e}_{35}, \overline{e}_{25}]) \\
&- \varphi([e_1, e_2, e_3, e_4, e_{14}, e_{24}, e_{34}, e_5, e_{35}, e_{1245}]) \\
&- \varphi([e_1, \overline{e}_{12}, e_2, \overline{e}_{13}, e_3, \overline{e}_{14}, e_4, u, \overline{e}_{45}, v]) \\
&+ \varphi([e_1, e_2, e_3, e_{14}, e_{24}, e_{34}, e_5, e_{15}, e_{35}, e_{1245}]) \\
&+ \varphi([e_1, e_2, e_3, e_4, e_{24}, e_{34}, e_{25}, e_{35}, e_{1245}, e_{1345}])\,.
\end{aligned}$$

where we denote the standard basis vectors in $\mathbb{R}^5$ by e_i, and we put $e_{ij} = e_i + e_j$, $\overline{e}_{ij} = -e_i + e_j$ and $e_{ijk\ell} = e_i + e_j + e_k + e_\ell$, as well as $u = e_5 - e_1 - e_4$ and $v = e_5 - e_2 - e_3$.

6. SPLITTING OFF THE VORONOÏ COMPLEX Vor_N FROM Vor_{N+1} FOR SMALL N

In this section, we will be concerned with $\Gamma = GL_N(\mathbb{Z})$ only and we adopt the notation $\Sigma_n(N) = \Sigma_n(GL_N(\mathbb{Z}))$ for the sets of representatives.

6.1. **Inflating well-rounded forms.** Let A be the symmetric matrix attached to a form h in C_N^*. Suppose the cell associated to A is *well-rounded*, i.e., its set of minimal vectors $S = S(A)$ spans the underlying vector space $\mathbb{R}^N$. Then we can associate to it a form $\tilde{h}$ with matrix $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & m(A) \end{pmatrix}$ in C_{N+1}^*, where $m(A)$ denotes the minimum positive value of A on $\mathbb{Z}^N$. The set $\tilde{S}$ of minimal vectors of $\tilde{A}$ contains the ones from S, each vector being extended by an $(N+1)$-th coordinate 0. Furthermore, $\tilde{S}$ contains the additional minimal vectors $\pm e_{N+1} = \pm(0, \ldots, 0, 1)$, and hence it spans $\mathbb{R}^{N+1}$, i.e., $\tilde{A}$ is well-rounded as well. In the following, we will call forms like $\tilde{A}$ as well as their associated cells *inflated*.

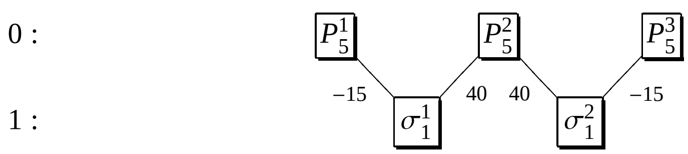
The stabilizer of h in $GL_N(\mathbb{Z})$ thereby embeds into the one of $\tilde{h}$ inside $GL_{N+1}(\mathbb{Z})$ (at least modulo $\pm\mathrm{Id}$) under the usual stabilization map.

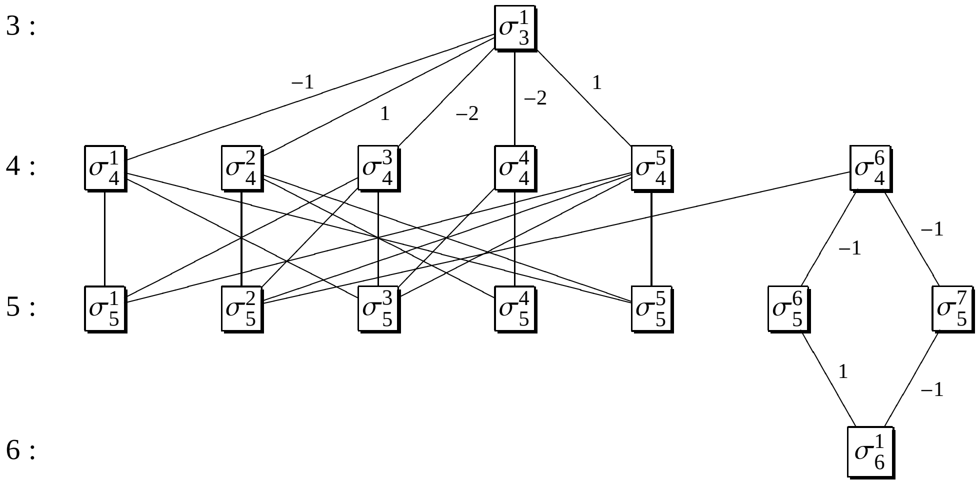
Note that, by iterating the same argument r times, A induces a well-rounded form also in $\Sigma_\bullet^\star(N+r)$ which, for $r \geqslant 2$, does not belong to $\Sigma_\bullet(N+r)$ since there is an obvious orientation-reversing automorphism of the inflated form, given by the permutation which swaps the last two coordinates.

6.2. **The case $N = 5$.**

Theorem 6.1. *The complex* $\mathrm{Vor}_{GL_5(\mathbb{Z})}$ *is isomorphic to a direct factor of* $\mathrm{Vor}_{GL_6(\mathbb{Z})}$, *with degrees shifted by 1.*

Proof. The Voronoï complex of $GL_5(\mathbb{Z})$ can be represented by the following weighted graph with levels





Here the nodes in line j (marked on the left) represent the elements in $\Sigma_{d(N)-j}(5)$, i.e. we have 3, 2, 0, 1, 6, 7 and 1 cells in codimensions 0, 1, 2, 3, 4, 5 and 6, respectively, and arrows show incidences of those cells, while numbers attached to arrows give the corresponding incidence multiplicities. Since entering the multiplicities relating codimensions 4 and 5 would make the graph rather unwieldy, we give them instead in terms of the matrix corresponding to the differential d_{10} connecting dimension 10 to 9 (columns refer, in this order, to $\sigma_5^1, \ldots, \sigma_5^7$, while rows refer to $\sigma_4^1, \ldots, \sigma_4^6$)

$$(4) \qquad \begin{pmatrix} -5 & 0 & -5 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 2 & -2 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

As is apparent from the picture, there are two connected components in that graph. The corresponding graph for $GL_6(\mathbb{Z})$ has three connected components, two of which are "isomorphic" (as weighted graphs with levels) to the one above for $GL_5(\mathbb{Z})$, except for a shift in codimension by 5 (e.g. codimension 0 cells in $\Sigma_\bullet(5)$ correspond to codimension 5 cells in $\Sigma_\bullet(6)$), i.e. a shift in dimension by 1.

In fact, it is possible, after appropriate coordinate transformations, to identify the minimal vectors (viewed up to sign) of any given cell in the two inflated components of $\Sigma_\bullet(6)$ alluded to above with the minimal vectors of another cell which is inflated from one in $\Sigma_\bullet(5)$, except precisely *one* minimal vector (up to sign) which is *fixed* under the stabilizer of the cell.

Let us illustrate this correspondence for the top-dimensional cell σ of the perfect form $P_5^1 \in \Sigma_{14}(5)$, also denoted $P(5,1)$ in [15] and D_5 in [19], with the list $m(P_5^1)$ of minimal vectors given already at the end of §5.2.

Using the algorithm described in §4.1, the corresponding inflated cell $\widetilde{\sigma}$ in $\Sigma_{15}(6)$ can be found to be, in terms of its 21 minimal vectors of the perfect form P_6^1 in Jaquet's notation (see [15] and §5.2 for the full list $m(P_6^1)$),

v_1	v_2	v_4	v_5	v_{10}	v_{12}	v_{13}	v_{14}	v_{16}	v_{17}	v_{18}	v_{22}	v_{24}	v_{25}	v_{26}	v_{27}	v_{29}	v_{33}	v_{34}	v_{35}	v_{36}
1	−1	0	−1	0	0	0	−1	1	1	0	1	1	0	1	0	0	1	0	0	1
0	1	−1	0	0	0	−1	0	1	0	1	1	0	1	0	1	0	0	1	0	1
0	0	1	1	0	−1	0	0	−1	0	0	−1	0	0	−1	−1	0	−1	−1	0	−1
0	0	0	0	1	0	0	0	−1	−1	−1	0	0	0	−1	−1	−1	0	0	0	−1
0	0	0	0	0	1	1	1	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	−1
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2

The transformation

$$\gamma = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 \end{pmatrix}$$

sends v_1 to $(0, 0, 0, 0, 0, 1)$ and sends each of the other vectors to the corresponding one of the form $(v, 0)$ where v is the corresponding minimal vector for P_5^1 (in the order given above).

One can verify that the other two perfect forms P_5^2 and P_5^3 (denoted by Voronoï A_5 and φ_2, respectively) give rise to a corresponding inflated cell in $\Sigma_{15}(6)$ in a similar way.

Concerning the cells of *positive* codimension in $\Sigma_\bullet(5)$, it turns out that these all have a representative which is a facet in σ. Furthermore, the matrix γ induces an isomorphism from the subcomplex of $\Sigma_\bullet(6)$ spanned by $\widetilde{\sigma}$ and all its facets to the complex obtained by inflation, as in §6.1 above, from the complex spanned by σ_5 and all its facets. Finally, one can verify that the cells attached to P_5^2 and P_5^3 are conjugate, after inflation, to cells in $\Sigma_{15}(6)$, and that the differentials for Vor_{GL_5} and Vor_{GL_6} agree on these. This ends the proof of the theorem. $\square$

6.3. **Other cases.** A similar situation holds for $\Sigma_\bullet(3)$ and $\Sigma_\bullet(4)$, but as $\Sigma_\bullet(3)$ consists of a single cell only, the picture is far less significant.

For $N = 4$, there is only one cell leftover in $\Sigma_\bullet(4)$, in fact in $\Sigma_6(4)$, and it is already inflated from $\Sigma_5(3)$. Hence its image in $\Sigma_7^\star(5)$ will allow an orientation reversing automorphism and hence will not show up in $\Sigma_7(5)$. This illustrates the remark at the end of 6.1.

Finally, for $N = 6$, the cells in the third component of the incidence graph for $GL_6(\mathbb{Z})$ mentioned in the proof of Theorem 6.1 above appear, in inflated form, in the Voronoï complex for $GL_7(\mathbb{Z})$ which inherits the homology of that component, since in the weighted graph of $GL_7(\mathbb{Z})$, which is connected, there is only one incidence of an inflated cell with a non-inflated one. Therefore we do not have a splitting in this case.

7. THE COHOMOLOGY OF MODULAR GROUPS

7.1. **Preliminaries.** Recall the following simple fact:

Lemma 7.1. *Assume that p is a prime and $g \in \mathrm{GL}_N(\mathbb{R})$ has order p. Then $p \leqslant N + 1$.*

Proof. The minimal polynomial of g is the cyclotomic polynomial $x^{p-1} + x^{p-2} + \cdots + 1$. By the Cayley-Hamilton theorem, this polynomial divides the characteristic polynomial of g. Therefore $p - 1 \leqslant N$. □

We shall also need the following result:

Lemma 7.2. *The action of* $\mathrm{GL}_N(\mathbb{R})$ *on the symmetric space* X_N *preserves its orientation if and only if N is odd.*

Proof. The subgroup $\mathrm{GL}_N(\mathbb{R})^+ \subset \mathrm{GL}_N(\mathbb{R})$ of elements with positive determinant is the connected component of the identity, therefore it preserves the orientation of X_N. Any $g \in \mathrm{GL}_N(\mathbb{R})$ which is not in $\mathrm{GL}_N(\mathbb{R})^+$ is the product of an element of $\mathrm{GL}_N(\mathbb{R})^+$ with the diagonal matrix $\varepsilon = \mathrm{diag}\,(-1, 1, \ldots, 1)$, so we just need to check when ε preserves the orientation of X_N. The tangent space TX_N of X_N at the origin consists of real symmetric matrices $m = (m_{ij})$ of trace zero. The action of ε is given by $m \cdot \varepsilon = \varepsilon^t\, m\, \varepsilon$ (cf. §2.1) and we get

$$(m \cdot \varepsilon)_{ij} = m_{ij}$$

unless $i = 1$ or $j = 1$ and $i \neq j$, in which case $(m \cdot \varepsilon)_{ij} = -m_{ij}$. Let δ_{ij} be the matrix with entry 1 in row i and column j, and zero elsewhere. A basis of TX_N consists of the matrices $\delta_{ij} + \delta_{ji}$, $i \neq j$, together with $N - 1$ diagonal matrices. For this basis, the action of ε maps $N - 1$ vectors v to their opposite $-v$ and fixes the other ones. The lemma follows. □

7.2. **Borel/Serre duality.** According to Borel and Serre ([6], Thm. 11.4.4 and Thm. 11.5.1), the group $\Gamma = SL_N(\mathbb{Z})$ or $GL_N(\mathbb{Z})$ is a virtual duality group with dualizing module

$$H^{v(N)}(\Gamma, \mathbb{Z}[\Gamma]) = \mathrm{St}_N \otimes \tilde{\mathbb{Z}}\,,$$

where $v(N) = N(N-1)/2$ is the virtual cohomological dimension of Γ and $\tilde{\mathbb{Z}}$ is the orientation module of X_N. It follows that there is a long exact sequence

$$\cdots \to H_n(\Gamma, \mathrm{St}_N) \to H^{v(N)-n}(\Gamma, \tilde{\mathbb{Z}}) \to \hat{H}^{v(N)-n}(\Gamma, \tilde{\mathbb{Z}}) \to H_{n-1}(\Gamma, \mathrm{St}_N) \to \cdots \tag{5}$$

where $\hat{H}^*$ is the Farrell cohomology of Γ [11]. From Lemma 7.1 and the Brown spectral sequence ([7], X (4.1)) we deduce that $\hat{H}^*(\Gamma, \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_{N+1}$. Therefore

$$H_n(\Gamma, \mathrm{St}_N) \equiv H^{v(N)-n}(\Gamma, \tilde{\mathbb{Z}})\,, \text{ modulo } \mathcal{S}_{N+1}. \tag{6}$$

When N is odd, then $\mathrm{GL}_N(\mathbb{Z})$ is the product of $\mathrm{SL}_N(\mathbb{Z})$ by $\mathbb{Z}/2$, therefore

$$H^m(\mathrm{GL}_N(\mathbb{Z}), \mathbb{Z}) \equiv H^m(\mathrm{SL}_N(\mathbb{Z}), \mathbb{Z})\,, \text{ modulo } \mathcal{S}_2.$$

When N is even, then the action of $\mathrm{GL}_N(\mathbb{Z})$ on $\tilde{\mathbb{Z}}$ is given by the sign of the determinant (see Lemma 7.2) and Shapiro's lemma gives

$$H^m(\mathrm{SL}_N(\mathbb{Z}), \mathbb{Z}) = H^m(\mathrm{GL}_N(\mathbb{Z}), M)\,, \tag{7}$$

with

$$M = \mathrm{Ind}_{\mathrm{SL}_N(\mathbb{Z})}^{\mathrm{GL}_N(\mathbb{Z})}\, \mathbb{Z} \equiv \mathbb{Z} \oplus \tilde{\mathbb{Z}}\,, \text{ modulo } \mathcal{S}_2.$$

7.3. **The cohomology of modular groups.** When $\Gamma = SL_N(\mathbb{Z})$ or $GL_N(\mathbb{Z})$, where $N \leqslant 7$, we know $H^m(\Gamma, \tilde{\mathbb{Z}})$ by combining (3) (end of §3.4), Theorem 4.3 and (6). As shown above, this allows us to compute the cohomology of Γ with trivial coefficients. The results are given in Theorem 7.3 below.

Theorem 7.3. (i) *Modulo $\mathcal{S}_5$ we have*

$$H^m(SL_5(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad m = 0, 5, \\ 0 & \text{otherwise}. \end{cases}$$

(ii) *Modulo $\mathcal{S}_7$ we have*

$$H^m(GL_6(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad m = 0, 5, 8, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$H^m(SL_6(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z}^2 & \text{if} \quad m = 5, \\ \mathbb{Z} & \text{if} \quad m = 0, 8, 9, 10, \\ 0 & \text{otherwise}. \end{cases}$$

(iii) *Modulo $\mathcal{S}_7$ we get that*

$$H^m(SL_7(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad m = 0, 5, 9, 14, 15, \\ 0 & \text{if} \quad 1 \leqslant m \leqslant 20. \end{cases}$$

For the proof of the final statement on integral cohomology (modulo $\mathcal{S}_7$) we use the fact that there are no primes $p > 7$ that divide the elementary divisors of the corresponding differentials or the order of the stabilizer of a cell in Σ_{27-m}.

***Remark* 7.4.** Morita asks in [21] whether the class of infinite order in $H^5(GL_5(\mathbb{Z}), \mathbb{Z})$ survives in the cohomology of the group of outer automorphisms of the free group of rank six.

***Remark* 7.5.** It was shown by A. Borel [5] that, for N large enough, $H^5(SL_N(\mathbb{Z}), \mathbb{Q})$ has dimension one. In view of Theorem 7.3 it is tempting to believe that the restriction map from $H^5(SL_N(\mathbb{Z}), \mathbb{Q})$ to $H^5(SL_5(\mathbb{Z}), \mathbb{Q})$ is an isomorphism. We have been unable to show that. An analogous statement holds for $H^9(SL_N(\mathbb{Z}), \mathbb{Q})$. Theorem 7.3 suggests that the non-trivial cohomology class already occurs when $N = 6$ and 7, i.e., in the “non-stable range".

8. Homology of modular groups with coefficients in the Steinberg module

In this section we compute some homology groups of $GL_N(\mathbb{Z})$ with coefficients the Steinberg module. Note that, when $N \geqslant 1$, the group $H_0(GL_N(\mathbb{Z}), St_N)$ vanishes [19].

Theorem 8.1. (i) *Modulo $\mathcal{S}_2$ we have*

$$H_3(GL_3(\mathbb{Z}), St_3) \cong \mathbb{Z} \tag{8}$$

and

$$H_3(GL_4(\mathbb{Z}), St_4) \cong \mathbb{Z}\,. \tag{9}$$

(ii) *The following groups lie in* $\mathcal{S}_2$*:*

$$\begin{array}{ll} H_4(GL_2(\mathbb{Z}), St_2)\,, & H_5(GL_2(\mathbb{Z}), St_2)\,, \\ H_4(GL_3(\mathbb{Z}), St_3)\,, & \\ H_2(GL_4(\mathbb{Z}), St_4)\,, & \\ H_1(GL_5(\mathbb{Z}), St_5)\,, & H_2(GL_5(\mathbb{Z}), St_5)\,, \\ H_1(GL_6(\mathbb{Z}), St_6)\,. & \end{array}$$

(iii) *The groups* $H_2(GL_6(\mathbb{Z}), St_6)$ *and* $H_1(GL_7(\mathbb{Z}), St_7)$ *lie in* $\mathcal{S}_5$.

In order to prepare for the proof, we first compute several terms in the spectral sequence E^1_{pq} of §3.2. This is done in five lemmas, dealing with $GL_4(\mathbb{Z})$, $GL_5(\mathbb{Z})$, $GL_6(\mathbb{Z})$ (separating the cases $p+q=6$ and $p+q=7$) and $GL_7(\mathbb{Z})$, respectively. We will show that the E^1 terms of the respective equivariant spectral sequences, in the desired ranges, are all zero modulo some torsion classes (mostly $\mathcal{S}_2$) which will allow us to deduce the claims. The general strategy is as follows: if G is the stabilizer of a cell, we will construct the maximal normal subgroup H of G which acts trivially on the cell. The quotient G/H will be in $\mathcal{S}_2$. Hence, using the Lyndon/Hochschild/Serre spectral sequence (denoted LHS in the remaining of the paper), the computation of the homology of G with coefficients in $\tilde{\mathbb{Z}}$ (i.e., $\mathbb{Z}$ endowed with the G-action on the cell) will be reduced to the computation of the homology of H with trivial coefficients. It will result that, in general, the corresponding homology groups lie in $\mathcal{S}_2$. We start by giving two general lemma, with straightforward proofs, that will be systematically used in our arguments.

Lemma 8.2. *Let* Γ *be a subgroup of* $GL_N(\mathbb{Z})$ *and let* σ *be a cell of* $\Sigma^\star_n(\Gamma)$*, for some* n*. Let* Γ_σ *be the stabilizer of the cell* σ *in* Γ*. Then there exists a normal subgroup* H *of* Γ_σ*, acting trivially on the cell* σ *and with quotient* Γ_σ/H *isomorphic to* $\mathbb{Z}/2$*.*

Proof. The action on the cell is given by η (see §3.1). It defines a morphism $\Gamma_\sigma \to \mathbb{Z}/2$ mapping γ to $\eta(\gamma\cdot\sigma, \sigma)$. We define H to be the kernel of this map. □

Lemma 8.3. *Consider a short exact sequence of finite groups*

$$1 \to H \to G \to Q \to 1\,.$$

Assume $Q \in \mathcal{S}_p$ *for some prime* p*. Let* M *be a* G*-module and* k *a positive integer. If* $H_i(H, M) \in \mathcal{S}_p$ *for all positive* $i \leqslant k$*, then* $H_k(G, M) \in \mathcal{S}_p$*.*

Now, we can compute the relevant parts of the equivariant spectral sequences of §3.2.

Lemma 8.4. *The terms* $E^1_{5,1}$*,* $E^1_{4,2}$*,* $E^2_{3,3}$*,* $E^1_{4,1}$ *and* $E^1_{3,2}$ *of the equivariant spectral sequence associated to* $\Gamma = GL_4(\mathbb{Z})$ *lie in* $\mathcal{S}_2$*.*

Proof.
- *Computation of* $E^1_{5,1}$. According to [19], Lemma 3.2, the set $\Sigma^\star_5(SL_4(\mathbb{Z}))$ consists of four cells, denoted σ^5_i ($i = 2,3,4,5$) in op.cit.

  The stabilizer of σ^5_5 in $PGL_4(\mathbb{Z})$ is isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_3$ (op.cit., p.121), each factor acting non-trivially on the orientation module of σ^5_5. It follows that $Stab(\sigma^5_5)$ contains a subgroup isomorphic to $\mathfrak{S}_3$ and preserving the orientation of σ^5_5. Therefore, modulo $\mathcal{S}_2$, we get

$$H_1(Stab(\sigma^5_5), \tilde{\mathbb{Z}}) = H_1(\mathfrak{S}_3, \mathbb{Z})\,.$$

From the exact sequence

$$1 \longrightarrow \mathbb{Z}/3 \longrightarrow \mathfrak{S}_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

we deduce that, modulo $\mathcal{S}_2$,

$$H_1(\mathfrak{S}_3,\mathbb{Z}) = H_0(\mathbb{Z}/2, H_1(\mathbb{Z}/3,\mathbb{Z})) = 0\,.$$

The stabilizer $Stab(\sigma_4^5)$ in $GL_4(\mathbb{Z})$ has order 32. Therefore its first homology group lies in $\mathcal{S}_2$.

The cell σ_3^5 of [19] (p.110) has its stabilizer (in $GL_4(\mathbb{Z})$) generated by the matrices

$$g_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 1\\ -1 & -1 & 0 & -1\\ 0 & 1 & 1 & 1\\ 1 & 0 & 0 & 0 \end{pmatrix},\quad g_{2,2} = \begin{pmatrix} 1 & 1 & 1 & 1\\ 0 & -1 & -2 & -2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix},\quad g_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix},\quad g_{2,4} = \begin{pmatrix} 1 & 0 & 0 & 0\\ -1 & -1 & -2 & -2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote by G_2 this group. It is of order $288 = 2^5 \cdot 3^2$. All the generators have a non-trivial action on the cell, except $g_{2,2}$. Let H_2 be the subgroup of G_2 generated by $g_{2,1}g_{2,3}$, $g_{2,2}$ and $g_{2,1}g_{2,4}$. By construction H_2 acts trivially on the cell. Using GAP, we can check that H_2 is normal in G_2 and the quotient G_2/H_2 is isomorphic to $\mathbb{Z}/2$. Furthermore, the derived subgroup of H_2 is isomorphic to $\mathbb{Z}/6\times\mathbb{Z}/3$ and its abelianization is isomorphic to $\mathbb{Z}/4\times\mathbb{Z}/2$. As a result $H_1(G_2,\tilde{\mathbb{Z}}) = H_0(\mathbb{Z}/2, H_1(H_2,\tilde{\mathbb{Z}})) = H_0(\mathbb{Z}/2, H_1(H_2;\mathbb{Z})) = 0$ mod $\mathcal{S}_2$.

The last cell to consider is σ_2^5. Let G_4 be the stabilizer of this cell. A set of generators of G_4 is given by the matrices

$$g_{4,1} = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix},\quad g_{4,2} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$g_{4,3} = \begin{pmatrix} 1 & 1 & 1 & 1\\ 0 & -1 & 0 & -1\\ 0 & 0 & -1 & -1\\ 0 & 0 & 0 & 1 \end{pmatrix},\quad g_{4,4} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -1 & -1 & -1 & -1 \end{pmatrix},\quad g_{4,5} = \begin{pmatrix} 1 & 0 & 0 & 0\\ -1 & 0 & -1 & -1\\ -1 & -1 & 0 & -1\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its order is $96 = 2^5 \cdot 3$. The group G_4 is isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2\times\mathfrak{S}_4$. Among the generators, only $g_{4,3}$ and $g_{4,5}$ have a non-trivial action. The subgroup generated by $g_{4,1}, g_{4,2}, g_{4,4}, g_{4,3}g_{4,5}$ is normal and isomorphic to $\mathbb{Z}/2\times\mathfrak{S}_4$. Its abelianization is isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2$. We deduce that $H_1(G_4,\tilde{\mathbb{Z}}) = H_0(\mathbb{Z}/2, H_1(\mathbb{Z}/2\times\mathfrak{S}_4,\mathbb{Z})) = 0 \mod \mathcal{S}_2$, and this ends the computations of $E^1_{5,1}$.

- *Computation of* $E^1_{4,2}$ *and* $E^1_{4,1}$. According to [19], Lemma 3.2, the set $\Sigma_4^\star(SL_4(\mathbb{Z}))$ consists of the three cells σ_2^4, σ_3^4 and σ_4^4. The stabilizer of σ_4^4 in $PGL_4(\mathbb{Z})$ is isomorphic to $\mathfrak{S}_5$ (op.cit., p.121). Modulo $\mathcal{S}_2$, the group $H_2(Stab(\sigma_4^4),\tilde{\mathbb{Z}})$ is thus a quotient of $H_2(\mathbb{Z}/5,\mathbb{Z})\oplus H_2(\mathbb{Z}/3,\mathbb{Z}) = 0$.

  Furthermore, the alternating subgroup $\mathfrak{A}_5 \subset \mathfrak{S}_5$ preserves the orientation of σ_4^4 ([19], Lemma 3.4), and it is equal to its commutator subgroup.

Therefore, modulo $\mathcal{S}_2$,

$$H_1(Stab(\sigma_4^4), \tilde{\mathbb{Z}}) = H_0(\mathbb{Z}/2, H_1(\mathfrak{A}_5, \mathbb{Z})) = 0\,.$$

- *Computation of* $E^2_{3,3}$. The only cell in $\Sigma_3^\star(GL_4(\mathbb{Z}))$ is the cell σ_3^3 of [19], Lemma 3.2. The action of $Stab(\sigma_3^3)$ on the orientation module is not trivial ([19], Lemma 3.3). According to [30], §3.2, we have

$$H_3(Stab(\sigma_3^3), \tilde{\mathbb{Z}}) \cong H_3(\mathfrak{A}_4, \mathbb{Z}) = \mathbb{Z}/3$$

modulo $\mathcal{S}_2$, and the differential

$$d^1\,:\; H_3(Stab(\sigma_3^4), \tilde{\mathbb{Z}}) \to H_3(Stab(\sigma_3^3), \tilde{\mathbb{Z}})$$

is surjective. Therefore $E^2_{3,3}$ lies in $\mathcal{S}_2$.
- *Computation of* $E^1_{3,2}$. Modulo $\mathcal{S}_2$, we get that $E^1_{3,2} = H_2(Stab(\sigma_3^3), \tilde{\mathbb{Z}})$ is a quotient of $H_2(\mathbb{Z}/3, \mathbb{Z}) = 0$.

□

Lemma 8.5. *The terms $E^1_{4,1}$, $E^1_{4,2}$, $E^1_{5,1}$ and $E^1_{6,0}$ of the equivariant spectral sequence associated to $\Gamma = GL_5(\mathbb{Z})$ are zero modulo $\mathcal{S}_2$.*

Proof.
- As none of the cells of $\Sigma_6^\star$ has its orientation preserved by the action of its stabilizer (see Fig.1), we have $E^1_{6,0} = 0 \mod \mathcal{S}_2$.

- Computation of $E^1_{5,1}$. We need to know the group $H_1(Stab_\Gamma(\sigma)\,, \tilde{\mathbb{Z}})$ for all five cells $\sigma \in \Sigma_5^\star$ (cf. Table 2). Up to equivalence under $GL_5(\mathbb{Z})$, these cells are contained in $\sigma(P_5^2)$, where P_5^2 is the perfect form of rank 5 described in [20], §6.4 (and mentioned above at the end of §6.2). We will denote these five cells by σ_i' with $i = 1, \ldots, 5$.

  Analyzing the cell σ_1'. First, let us describe σ_1'. The 15 minimal vectors of P_5^2 are given below, together with their label:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	1	0	0	0	1	0	0	1	1	0	1
0	1	0	0	0	1	0	0	0	1	0	1	0	1	1
0	0	1	0	0	0	1	0	0	0	1	0	1	1	1
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

  The vertices of the cell σ_1' are the forms $\hat{v}$ where v is one of the vectors labelled by 1, 2, 3, 4, 5 and 8. Set $G_1' = Stab_\Gamma(\sigma_1')$. A set of generators of G_1' is given by the following six matrices of $GL_5(\mathbb{Z})$, of respective order 6, 2, 2, 2, 2, 6 :

$$g'_{1,1} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \; g'_{1,2} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \; g'_{1,3} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g'_{1,4} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; g'_{1,5} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; g'_{1,6} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order of G'_1 is $576 = 2^6 \cdot 3^2$. Thus, *a priori*, we could expect some 3–torsion in the homology of this group. Only $g'_{1,2}$ and $g'_{1,6}$ have a trivial action on the cells. Using GAP [12], we get that the group G'_1 is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathfrak{S}_3 \times \mathfrak{S}_4$. Let H'_1 be the subgroup of G'_1 generated by $g'_{1,2}, g'_{1,6}, g'_{1,1}g'_{1,3}, g'_{1,1}g'_{1,4}, g'_{1,1}g'_{1,5}$. By construction this subgroup has trivial action on the cell. It is normal and has order 288. We then have $G'_1/H'_1 = \mathbb{Z}/2$. Furthermore, the derived subgroup of H'_1 is isomorphic to $\mathbb{Z}/3 \times \mathfrak{A}_4$ and the quotient $H'_1/[H'_1, H'_1]$ is isomorphic to the product of three copies of $\mathbb{Z}/2$. Thus the first homology group of H'_1 with trivial coefficients is zero modulo $\mathcal{S}_2$. By Lemma 8.3, we get

$$H_1(G'_1, \tilde{\mathbb{Z}}) = H_0(\mathbb{Z}/2, H_1(H'_1; \mathbb{Z})) = 0 \mod \mathcal{S}_2 .$$

Analyzing the cell σ'_2. The cell σ'_2 is given by the vectors labelled by 1, 2, 3, 5, 6 and 8. Denote its stabilizer by G'_2. A set of generators of G'_2 consists of the following six matrices of $GL_5(\mathbb{Z})$, of respective order 2, 4, 4, 4, 2, 2:

$$g'_{2,1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \; g'_{2,2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \; g'_{2,3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$g'_{2,4} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; g'_{2,5} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; g'_{2,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order of the stabilizer is $384 = 2^7 \cdot 3$.
Using GAP we get that G'_2 is isomorphic to $\mathbb{Z}/2 \times \mathfrak{S}_4 \times D_8$. The generators $g'_{2,1}$, $g'_{2,2}$ and $g'_{2,6}$ act trivially on the cell. Consider the subgroup H'_2 of G'_2 generated by $g'_{2,1}$, $g'_{2,2}$, $g'_{2,6}$, $g'_{2,3}g'_{2,4}$ and $g'_{2,3}g'_{2,5}$. This subgroup is normal and acts trivially on the cell. Its order is 192, thus the quotient G'_2/H'_2 is of order 2. We can check with GAP that the abelianization of H'_2 is isomorphic to $(\mathbb{Z}/2)^3$. We deduce, by Lemma 8.3, that

$$H_1(G'_2; \tilde{\mathbb{Z}}) = H_0(\mathbb{Z}/2; H_1(H'_2; \mathbb{Z})) = 0 \mod \mathcal{S}_2.$$

Analyzing the cell σ'_3. The cell σ'_3 is given by the vectors labelled by 2, 3, 5, 6, 8, 9. Denote its stabilizer by G'_3. A set of generators of G'_3 consists of the following three matrices of $GL_5(\mathbb{Z})$, of respective order 2, 10, 4:

$$g'_{3,1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \; g'_{3,2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix}, \; g'_{3,3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The order of the stabilizer is $480 = 2^5 \cdot 3 \cdot 5$. Using GAP, we see that the group G'_3 is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathfrak{S}_5$. Among the generators only $g'_{3,3}$ has a non-trivial action on the cell. Let us consider the subgroup of G'_3, denoted H'_3, generated by $g'_{3,1}$, $g'_{3,2}$ and ${g'_{3,3}}^2$. The subgroup H'_3 acts trivially on the cell, it is normal and isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathfrak{A}_5$. Thus $G'_3/H'_3 = \mathbb{Z}/2$ and, as $\mathfrak{A}_n$ is perfect for $n \geqslant 5$, the abelianization of H'_3 is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. By Lemma 8.3, we get

$$H_1(G'_3; \tilde{\mathbb{Z}}) = H_0(\mathbb{Z}/2; H_1(H'_3; \mathbb{Z})) = 0 \mod \mathcal{S}_2 .$$

Analyzing the cell σ'_4. The cell σ'_4 is given by the vectors labelled by 1, 2, 3, 7, 11, 12. A set of generators of its stabilizer is given by the following two matrices of $GL_5(\mathbb{Z})$, of respective order 6, 2:

$$g'_{4,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \; g'_{4,2} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

The order of the stabilizer is $240 = 2^4 \cdot 3 \cdot 5$. Denote G'_4 this group, which is isomorphic to $\mathbb{Z}/2 \times \mathfrak{S}_5$. Only the generator $g'_{4,1}$ acts non-trivially on the cell. Let H'_4 be the subgroup generated by ${g'_{4,1}}^2$ and $g'_{4,2}$. This subgroup is normal and isomorphic to $\mathbb{Z}/2 \times \mathfrak{A}_5$. So, as above, we get $H_1(G'_4; \tilde{\mathbb{Z}}) = 0 \mod \mathcal{S}_2$.

Analyzing the cell σ'_5. The cell σ'_5 is given by the vectors labelled by 2, 3, 5, 7, 9, 10. A set of generators of its stabilizer, denoted G'_5, is given by the following three matrices of $GL_5(\mathbb{Z})$, of respective order 6, 6, 2:

$$g'_{5,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \; g'_{5,2} = \begin{pmatrix} -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \; g'_{5,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

The stabilizer is of order $1440 = 2^5 \cdot 3^2 \cdot 5$. The group is isomorphic to $\mathbb{Z}/2 \times \mathfrak{S}_6$. Among the generators only $g'_{5,3}$ has a non-trivial action. The subgroup generated by $g'_{5,1}$ and $g'_{5,2}$ is normal, it acts trivially on σ'_5 and is isomorphic to $\mathbb{Z}/2 \times \mathfrak{A}_6$. So we get $H_1(G'_5; \tilde{\mathbb{Z}}) = 0 \mod \mathcal{S}_2$.

- Computation of $E^1_{4,2}$. The set $\Sigma^\star_4$ consists of two cells contained (up to equivalence) in $\sigma(P^2_5)$. We will denote those cells by τ'_i with $i = 1, 2$.

Analyzing the cell τ'_1 and the cell τ'_2. The cell τ'_1 is given by the vectors labelled by 1, 2, 3, 4 and 8 in $m(P^2_5)$. A set of generators of its stabilizer is given by the following three matrices of $GL_5(\mathbb{Z})$, of respective order 2, 6, 2:

$$t_{1,1}=\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix},\ t_{1,2}=\begin{pmatrix}0&-1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\1&0&0&0&0\end{pmatrix},\ t_{1,3}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-1\end{pmatrix}.$$

The order of the stabilizer is $3840 = 2^8\cdot 3\cdot 5$. Furthermore, only $t_{1,2}$ has a non-trivial action on the cell.
The cell τ'_2 is given by the vectors 1, 2, 7, 11 and 12. A set of generators of its stabilizer is given by the following three matrices of $GL_5(\mathbb{Z})$, of respective order 6, 4, 2:

$$t_{2,1}=\begin{pmatrix}0&0&0&0&1\\0&0&0&-1&1\\1&-1&0&0&0\\0&-1&0&0&1\\1&0&1&-1&0\end{pmatrix},\ t_{2,2}=\begin{pmatrix}-1&0&-1&0&1\\0&0&0&-1&1\\0&1&0&0&0\\0&0&-1&0&1\\0&1&0&-1&0\end{pmatrix},\ t_{2,3}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&-1\\0&0&0&1&0\\0&0&-1&1&0\end{pmatrix}.$$

The order of the stabilizer is $3840 = 2^8\cdot 3\cdot 5$.

We need to analyze the first and second homology groups of the stabilizers of τ'_1 and τ'_2. Using GAP, it is possible to show that these two stabilizers are isomorphic. Set

$$h_{1,1}=t_{1,2}^{-2}t_{1,1}t_{1,2}t_{1,1}t_{1,2}^{-1}t_{1,1}t_{1,2}^{2}t_{1,1}t_{1,2}t_{1,1}=\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\-1&0&0&0&0\end{pmatrix},$$

$$h_{1,2}=t_{1,1}t_{1,3}t_{1,2}^{-1}t_{1,1}t_{1,2}^{-1}t_{1,1}t_{1,2}t_{1,1}=\begin{pmatrix}0&-1&0&0&0\\1&0&0&0&0\\0&0&0&0&-1\\0&0&-1&0&0\\0&0&0&-1&0\end{pmatrix},$$

$$h_{2,1}=t_{2,3}t_{2,1}t_{2,2}^{-1}t_{2,1}^{2}t_{2,2}t_{2,1}^{2}=\begin{pmatrix}1&0&0&0&0\\0&0&-1&1&0\\0&1&1&0&-1\\0&0&0&1&0\\0&1&0&0&0\end{pmatrix},$$

$$h_{2,2}=t_{2,3}t_{2,1}^{-1}t_{2,3}t_{2,2}^{-1}t_{2,1}^{2}=\begin{pmatrix}0&-1&-1&1&0\\1&0&0&0&-1\\0&0&0&-1&0\\0&0&0&0&-1\\0&0&-1&0&0\end{pmatrix}.$$

Then the group generated by $h_{1,1}$ and $h_{1,2}$ (resp. $h_{2,1}$ and $h_{2,2}$) is isomorphic to $Stab(\tau'_1)$ (resp. $Stab(\tau'_2)$). We can check that the mapping sending $h_{1,1}$ resp. $h_{1,2}$) to $h_{2,1}$ (resp. $h_{2,2}$) defines a group isomorphism. Hence it suffices to consider τ'_1. Let H be the subgroup generated by $t_{1,1}$, $t_{1,3}$, $(t_{1,1}t_{1,2})^2$ and $(t_{1,3}t_{1,2})^2$. Then H is normal, of order 1920 and it acts trivially on τ'_1. Using

GAP, we can check that its abelianization is isomorphic to $\mathbb{Z}/2$. Using a composition series for H, we get a short exact sequence

$$0 \to (\mathbb{Z}/2)^5 \to H \to \mathfrak{A}_5 \to 1\,.$$

The homology of $(\mathbb{Z}/2)^5$ is trivial modulo $\mathcal{S}_2$ except

$$H_0((\mathbb{Z}/2)^5\,,\tilde{\mathbb{Z}}) = H_0((\mathbb{Z}/2)^5\,,\mathbb{Z}) = \mathbb{Z}.$$

But as $\mathfrak{A}_5$ is simple, we deduce that it acts trivially on $H_0((\mathbb{Z}/2)^5\,,\tilde{\mathbb{Z}})$. Since $H_i(\mathfrak{A}_5, \mathbb{Z})$ lies in $\mathcal{S}_2$ for $i = 1, 2$ [32], we deduce that

$$H_i(\mathfrak{A}_5, H_j((\mathbb{Z}/2)^5\,,\tilde{\mathbb{Z}})) = 0 \mod \mathcal{S}_2\,, \text{with } i + j = 1, 2.$$

Using the LHS spectral sequence associated to the above exact sequence, we get $H_i(H, \tilde{\mathbb{Z}}) = 0$ modulo $\mathcal{S}_2$ and by Lemma 8.3, $H_i(Stab(\tau'_1), \tilde{\mathbb{Z}}) = 0$ mod $\mathcal{S}_2$ for $i = 1, 2$.

□

Lemma 8.6. *The terms $E^1_{5,1}$ and $E^1_{6,0}$ of the equivariant spectral sequence associated to $\Gamma = GL_6(\mathbb{Z})$ are zero modulo $\mathcal{S}_2$.*

Proof. The claim that $E^1_{6,0}$ is zero modulo $\mathcal{S}_2$ is again a consequence of the fact that none of the cells of $\Sigma^\star_6(GL_6(\mathbb{Z}))$ has its orientation preserved by the action of its stabilizer. It remains to show that $E^1_{5,1}$ is zero modulo $\mathcal{S}_2$.
From our computations (cf. Fig.1), we know that $\Sigma^\star_5(GL_6(\mathbb{Z}))$ has three cell representatives which can be chosen inside $\sigma(P^1_6)$. We will denote these three cells by τ_i $(i = 1, 2, 3)$. Here is the ordered list of minimal vectors of P^1_6 that we shall use:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	−1	0	0	−1	0	0	0	−1	0	0	0	0	−1	0	1	1	0
0	1	1	−1	0	0	0	−1	0	0	0	0	−1	0	0	1	0	1
0	0	0	1	1	1	−1	0	0	0	0	−1	0	0	0	−1	0	0
0	0	0	0	0	0	1	1	1	1	−1	0	0	0	0	−1	−1	−1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	−1	−1	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	1
1	0	1	1	0	0	1	0	1	1	0	0	1	0	0	1	0	1
0	−1	−1	−1	0	0	0	−1	−1	−1	0	0	0	−1	−1	−1	0	−1
−1	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	0	0	0	0	−1
−1	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0	−1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2

Analyzing the cell τ_1. The cell τ_1 can represented by the vectors 1, 15, 24, 25, 31, 34. Set $G_1 = Stab_\Gamma(\tau_1)$. A set of generators of G_1 consists of the following four matrices of $GL_6(\mathbb{Z})$, of respective order 4, 6, 4, 2:

$$g_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & -1 & -2 \\ 0 & -1 & 0 & 0 & 1 & 2 \end{pmatrix}, \quad g_{1,2} = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_{1,3} = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & -2 \\ 1 & -1 & -1 & -2 & 0 & 0 \end{pmatrix}, \quad g_{1,4} = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order of G_1 is $46080 = 2^{10} \cdot 3^2 \cdot 5$. Only $g_{1,3}$ has a non-trivial action on the cell. Consider the subgroup H_1 of G_1 generated by $g_{1,1}, g_{1,2}, g_{1,4}$ and $(g_{1,1}g_{1,3})^2$. Then by construction, this subgroup acts trivially on the cell. Using GAP, we can check that G_1 is isomorphic to $GM_6(\mathbb{Z})$, the subgroup of monomial matrices of $GL_6(\mathbb{Z})$ (semi-direct product of $\mathfrak{S}_6$ and $\{\pm 1\}^6$), and H_1 is normal, isomorphic to the semi-direct product of $\mathfrak{A}_6$ and $\{\pm 1\}^6$. Thus the quotient G_1/H_1 is isomorphic to $\mathbb{Z}/2$. Then, by the computation of the abelianization of semi-direct products, we get that $H_1/[H_1, H_1] \cong \mathbb{Z}/2$. We deduce that $H_1(H_1, \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_2$ and by Lemma 8.3, we conclude that $H_1(G_1, \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_2$.

Analyzing the cell τ_2. The cell τ_2 is given by the vectors 15, 24, 25, 28, 29, 34. Set $G_2 = Stab_\Gamma(\tau_2)$. A set of generators of G_2 consists of the following four matrices of $GL_6(\mathbb{Z})$, of respective order 4, 6, 2, 2:

$$g_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}, \quad g_{2,2} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & -1 & -1 \end{pmatrix},$$

$$g_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 & 0 & 1 \end{pmatrix}, \quad g_{2,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order of G_2 is $7680 = 2^9 \cdot 3 \cdot 5$. Only the generators $g_{2,3}$ and $g_{2,4}$ have a non-trivial action on the cell. Denote by H_2 the subgroup of G_2 generated by $g_{2,1}^2, g_{2,2}^2, g_{2,3}$ and $g_{2,4}$. Then H_2 acts trivially on the cell and can be checked, using GAP, to be normal and of order 3840. Furthermore, its abelianization is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. Using Lemma 8.3 as above, we deduce that $H_1(G_2, \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_2$.

Analyzing the cell τ_3. The cell τ_3 is given by the vectors 2, 15, 25, 28, 29, 34. Set $G_3 = Stab_\Gamma(\tau_3)$. A set of generators of G_3 consists of the following three matrices of $GL_6(\mathbb{Z})$, of respective order 6, 6, 2:

$$g_{3,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & -2 \end{pmatrix},\ g_{3,2} = \begin{pmatrix} -1 & -1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix},$$

$$g_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order of G_3 is $46080 = 2^{10} \cdot 3^2 \cdot 5$ and it is isomorphic to G_1. Only $g_{3,3}$ has a trivial action on the cell. The subgroup of G_3 generated by $g_{3,1}^2, g_{3,2}^2, g_{3,1}g_{3,2}$ and $g_{3,3}$ acts trivially on the cell, and using GAP, we can check that it is isomorphic to H_1. As a result, we conclude that $H_1(G_3, \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_2$. As all the terms of $E^1_{5,1}$ lies in $\mathcal{S}_2$, the lemma is proved. □

Lemma 8.7. *The terms $E^1_{5,2}$, $E^1_{6,1}$ and $E^1_{7,0}$ of the equivariant spectral sequence associated to $\Gamma = GL_6(\mathbb{Z})$ are zero modulo $\mathcal{S}_5$.*

Proof. Looking at the table of representatives for $\Sigma^\star_p(\Gamma)$ (cf. Figure 1), we see that there are three 5−cells, ten 6−cells and twenty-eight 7−cells. None of the 7-cells has its orientation preserved by its stabilizer. Thus $H_0(Stab_\Gamma(\sigma), \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_2$ for all $\sigma \in \Sigma^\star_7(\Gamma)$. Among the 6−cells, only one has a stabilizer with 7-torsion. It is the cell given by the minimal vectors

1	3	5	8	12	16	17
1	0	−1	0	0	0	0
0	1	0	−1	0	0	0
0	0	1	0	−1	0	0
0	0	0	1	0	0	−1
0	0	0	0	1	−1	0
0	0	0	0	0	1	1

from P^7_6. We will denote by G_0 its stabilizer. It is generated by the following matrices:

$$g_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},\ g_{0,2} = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$g_{0,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

all of which have a non-trivial action on the cell. The order of G_0 is $10080 = 2^5 \times 3^2 \times 5 \times 7$. From a composition series of G_0, we can deduce the exact sequence

$$1 \to H \to G_0 \to \mathbb{Z}/2 \to 1$$

where $H \cong \mathfrak{A}_7 \times \mathbb{Z}/2$ and H is generated by $g_{0,1}^2$, $g_{0,2}^2$ and $-\mathrm{Id}$. Hence the action of H on the cell is trivial. Furthermore, the quotient $H/[H,H]$ is isomorphic to $\mathbb{Z}/2$. From the previous data, we deduce by a spectral sequence argument that $H_1(G_0, \tilde{\mathbb{Z}}) = 0 \mod \mathcal{S}_2$. Finally, among the 5−cells, none of them has a stabilizer with 7-torsion. Lemma 8.7 follows. □

Lemma 8.8. *The terms $E^1_{6,1}$ and $E^1_{7,0}$ of the equivariant spectral sequence associated to $\Gamma = GL_7(\mathbb{Z})$ are zero modulo $\mathcal{S}_5$.*

Proof. Looking at the table of representative for $\Sigma_p^\star(\Gamma)$ (cf. Figure 2), we see that there are twenty-eight 7−cells, none of them having its orientation preserved by the action of its stabilizer. As a result, we can deduce that $E^1_{0,7} = 0 \mod \mathcal{S}_2$. Among the six 6-cells, only three have a stabilizer of order divisible by 7.They are the ones to investigate.

1. The first cell is given by the following seven minimal vectors

1	2	3	4	5	6	7
1	1	1	1	1	2	1
1	1	1	1	2	1	1
−1	−1	−1	0	−1	−1	−1
−1	−1	0	−1	−1	−1	−1
−1	0	−1	−1	−1	−1	−1
0	−1	−1	−1	−1	−1	−1
2	2	2	2	2	2	2

of P_7^2. A set of generators for its stabilizer, that we will denote by G_1, consists of the following matrices

$$g_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_{1,2} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$g_{1,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_{1,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_{1,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_{1,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The group G_1 is of order $10080 = 2^5 \times 3^2 \times 5 \times 7$. The generators $g_{1,1}$, $g_{1,2}$, $g_{1,5}$ and $g_{1,6}$ have a non-trivial action on the cell. A composition series of G_1 is given by

$$1 \triangleleft \mathfrak{A}_7 \triangleleft \mathfrak{S}_7 \triangleleft G_1 ,$$

with $G_1/\mathfrak{S}_7 \cong \mathbb{Z}/2$. The group $\mathfrak{A}_7$ is generated by $(g_{1,1}g_{1,2})^2$ and $g_{1,3}$, and $\mathfrak{S}_7$ is generated by $(g_{1,1}g_{1,2})^2$, $g_{1,3}$ and $g_{1,1}$. Using these generators, we deduce that the action of $\mathfrak{A}_7$ on the cell is trivial, while the one of $\mathfrak{S}_7$ is not. There are two spectral sequences:

$$H_i(\mathbb{Z}/2; H_j(\mathfrak{S}_7; \tilde{\mathbb{Z}})) \Longrightarrow H_{i+j}(G_1; \tilde{\mathbb{Z}}),$$
$$H_i(\mathbb{Z}/2; H_j(\mathfrak{A}_7; \tilde{\mathbb{Z}})) \Longrightarrow H_{i+j}(\mathfrak{S}_7; \tilde{\mathbb{Z}}).$$

The action of $\mathfrak{A}_7$ is trivial and this group is perfect, so we get $H_1(\mathfrak{S}_7; \tilde{\mathbb{Z}}) \cong \mathbb{Z}/2$. We deduce that $H_1(G_1; \tilde{\mathbb{Z}}) = 0 \mod \mathcal{S}_2$.

2. The second cell is given by the minimal vectors

1	2	3	5	8	12	23
−1	−1	0	0	0	0	0
−1	0	−1	0	0	0	0
−1	0	0	−1	0	0	0
−1	0	0	0	−1	0	0
−1	0	0	0	0	−1	0
−1	0	0	0	0	0	1
3	1	1	1	1	1	0

from P_7^{12}. We will denote its stabilizer by G_2, which is of order $645120 = 2^{11} \times 3^2 \times 5 \times 7$. A set of generators for G_2 is given by

$$g_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & -3 & -1 \end{pmatrix}, \quad g_{2,2} = \begin{pmatrix} -1 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 1 & -1 & 1 & 1 & 0 \end{pmatrix},$$

$$g_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Only $g_{2,2}$ has a non-trivial action on the cell. Using a composition series for G_2, we get the exact sequence

$$1 \to H \to G_2 \to \mathbb{Z}/2 \to 1$$

Using GAP, we can show that H is generated by $g_{2,1}$, $g_{2,3}$ and $g_{2,2}^2$. It follows that the action of H on the cell is trivial. We have the following spectral sequence

$$'E^2_{i,j} = H_i(\mathbb{Z}/2; H_j(H; \tilde{\mathbb{Z}})) \Longrightarrow H_{i+j}(G_2; \tilde{\mathbb{Z}}).$$

Furthermore, the group $H/[H, H]$ is isomorphic to $\mathbb{Z}/2$. As a result, we get that $H_1(G_2; \tilde{\mathbb{Z}}) = 0 \mod \mathcal{S}_2$.

3. The last cell is given by the minimal vectors

1	2	3	4	5	6	7
0	−1	0	0	0	0	0
0	0	−1	0	0	0	0
0	0	0	−1	0	0	0
0	0	0	0	−1	0	0
0	0	0	0	0	−1	0
0	0	0	0	0	0	−1
1	1	1	1	1	1	1

from P_7^{33}. We will denote its stabilizer by G_3. Its order is $645120 = 2^{11}\times 3^2\times 5\times 7$. The group G_3 is spanned by the following six matrices:

$$g_{3,1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad g_{3,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_{3,3} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 & -2 & -2 & -1 \end{pmatrix}, \quad g_{3,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_{3,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_{3,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & -2 & -2 & -2 & -2 & -1 \end{pmatrix}.$$

Only the generators $g_{3,2}$ and $g_{3,3}$ have a non-trivial action on the cell. The groups G_2 and G_3 turn out to be isomorphic. Hence the previous arguments will apply to G_3.

We have an exact sequence

$$1 \to H' \to G_3 \to \mathbb{Z}/2 \to 1\,,$$

where H' is generated by $g_{3,1}$, $g_{3,4}$, $g_{3,5}$ and $g_{3,6}$, together with the product $g_{3,2}g_{3,3}$. As a result, the action of H' on the cell is trivial. Moreover, the quotient $H'/[H', H']$ is isomorphic to $\mathbb{Z}/2$ and we deduce that $H_1(G_3; \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_2$. □

Now we are ready to complete the proof of Theorem 8.1:

Proof. (of Theorem 8.1).

- In rank 2, we shall prove that $H_4(GL_2(\mathbb{Z}), St_2)$ and $H_5(GL_2(\mathbb{Z}), St_2)$ lie in $\mathcal{S}_2$. Let $\tilde{\mathbb{Z}}$ be the orientation module of the symmetric space X_2 and let $\hat{H}^*(GL_2(\mathbb{Z}), \tilde{\mathbb{Z}})$ be the Farrell cohomology of $GL_2(\mathbb{Z})$ [11]. From (5) in §7.2 it follows that

$$H_4(GL_2(\mathbb{Z}), St_2) \cong \hat{H}^{-3}(GL_2(\mathbb{Z}), \tilde{\mathbb{Z}})\,,$$

and

$$H_5(GL_2(\mathbb{Z}), St_2) \cong \hat{H}^{-4}(GL_2(\mathbb{Z}), \tilde{\mathbb{Z}})\,.$$

As to the first claim, since the only 3-group contained in $GL_2(\mathbb{Z})$ is, up to conjugation, $\mathbb{Z}/3$, we get

$$\hat{H}^{-5}(GL_2(\mathbb{Z}), \tilde{\mathbb{Z}}) \subset \hat{H}^{-5}(\mathbb{Z}/3, \tilde{\mathbb{Z}}) = 0\,.$$

As to the second claim, $GL_2(\mathbb{Z})$ is an extension

$$1 \longrightarrow SL_2(\mathbb{Z}) \longrightarrow GL_2(\mathbb{Z}) \longrightarrow \Delta \longrightarrow 1$$

with $\Delta = \mathbb{Z}/2$, and $SL_2(\mathbb{Z})$ is the amalgamated product of $\mathbb{Z}/4$ and $\mathbb{Z}/6$ along $\mathbb{Z}/2$ (see [29]). Therefore

$$\hat{H}^{-4}(SL_2(\mathbb{Z}), \mathbb{Z}) = \hat{H}^{-4}(GL_2(\mathbb{Z}), \mathbb{Z}) \oplus \hat{H}^{-4}(GL_2(\mathbb{Z}), \tilde{\mathbb{Z}})$$

(see §7.2) and, modulo $\mathcal{S}_2$,

$$\hat{H}^{-4}(SL_2(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}/3\,.$$

Let β be a generator of $\hat{H}^{-2}(SL_2(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}/3$. Since $\hat{H}^{-4}(SL_2(\mathbb{Z}), \mathbb{Z})$ is spanned by β^2, the action of Δ on this group is trivial. Therefore

$$\hat{H}^{-4}(GL_2(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}/3$$

modulo $\mathcal{S}_2$ and $\hat{H}^{-4}(SL_2(\mathbb{Z}), \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_2$.

- In rank 3, we know from [31], Thm. 5(iii), that $H_3(GL_3(\mathbb{Z}), St_3) \cong \mathbb{Z}$ modulo $\mathcal{S}_2$. Moreover, from op.cit., Thm. 5(ii), we have

$$H_4(GL_3(\mathbb{Z}), St_3) \cong \hat{H}^{-2}(GL_3(\mathbb{Z}), \mathbb{Z})\,.$$

where $\hat{H}^*$ denotes the Farrell cohomology, and from op.cit., Corollary (i) on p.9, we know that

$$\hat{H}^{-2}(GL_3(\mathbb{Z}), \mathbb{Z}) \quad \text{lies in } \mathcal{S}_2\,.$$

- In rank 4, we know from [19], Lemma 3.3, that $\Sigma_p^\star$ is empty when $p < 3$, hence $E^1_{p,q} = 0$ when $p < 3$. We proved in Lemma 8.4 that $E^2_{p,q}$ lies in $\mathcal{S}_2$ when $q > 0$ and $p + q = 5$ or $p + q = 6$. According to [19], Proposition 3.1, $E^2_{0,6} \cong \mathbb{Z}$ and $E^2_{0,5} = 0$ modulo $\mathcal{S}_2$. Therefore, modulo $\mathcal{S}_2$, $H_3(GL_4(\mathbb{Z}), St_4) \cong \mathbb{Z}$ and $H_2(GL_4(\mathbb{Z}), St_4) = 0$.

- In rank 5, we see in Fig.1 that $\Sigma_p^\star$ is empty when $p < 4$. Therefore

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p^\star} H_q(\Gamma_\sigma, \tilde{\mathbb{Z}})$$

vanishes if $p < 4$. On the other hand, since $\Sigma_5(GL_5(\mathbb{Z}))$ is empty, the group $E^1_{5,0}$ lies in $\mathcal{S}_2$. We proved in Lemma 8.5 that $E^1_{4,1}$ is in $\mathcal{S}_2$. Therefore $E^1_{p,q}$ lies in $\mathcal{S}_2$ when $N = 5$ and $p + q = 5$, hence $H_1(GL_5(\mathbb{Z}), St_5)$ lies in $\mathcal{S}_2$.

Similarly, we know from Lemma 8.5 that $E^1_{p,q}$ is in $\mathcal{S}_2$ when $p \geqslant 4$ and $p + q = 6$. Therefore $H_2(GL_5(\mathbb{Z}), St_5)$ lies in $\mathcal{S}_2$.

- In rank 6, $\Sigma_p^\star$ is empty and $E^1_{p,q} = 0$ when $p < 5$.
  Lemma 8.6 shows that also $E^1_{p,q} = 0$ when $p \geqslant 5$ and $p + q = 6$. Therefore $H_1(GL_6(\mathbb{Z}), St_6)$ lies in $\mathcal{S}_2$.
  When $p + q = 7$ and $p \geqslant 5$, Lemma 8.7 shows that $E^1_{p,q}$ lies in $\mathcal{S}_5$. Therefore $H_2(GL_6(\mathbb{Z}), St_6)$ is in $\mathcal{S}_5$.

- In rank 7, $\Sigma_p^\star$ is empty and $E^1_{p,q} = 0$ when $p < 6$ (see Fig.1). On the other hand, we know from Lemma 8.8 that $E^1_{6,1}$ and $E^1_{7,0}$ lie in $\mathcal{S}_5$. Therefore $H_1(GL_7(\mathbb{Z}), St_7)$ lies in $\mathcal{S}_5$.

□

9. APPLICATION TO K-THEORY

The homology of the general linear group with coefficients in the Steinberg module can also be used to compute the K-theory of $\mathbb{Z}$. Let $P(\mathbb{Z})$ (resp. $P_N(\mathbb{Z})$) be the exact category of free $\mathbb{Z}$-modules of finite rank (resp. of rank at most N), let Q (resp. Q_N) be the category obtained from $P(\mathbb{Z})$ (resp. $P_N(\mathbb{Z})$) by the Q-construction [25], and let BQ (resp. BQ_N) be its classifying space. A definition of higher K-theory [25] is

$$K_m(\mathbb{Z}) = \pi_{m+1}(BQ)\,, \quad m \geqslant 0\,.$$

On the other hand, Quillen proved in [26] that there are long exact sequences

$$\begin{aligned}(10)\quad \cdots \to H_m(BQ_{N-1}, \mathbb{Z}) \to H_m(BQ_N, \mathbb{Z}) \to H_{m-N}(\mathrm{GL}_N(\mathbb{Z}), \mathrm{St}_N) \\ \to H_{m-1}(BQ_{N-1}, \mathbb{Z}) \to \cdots\,,\end{aligned}$$

and, according to Lee and Szczarba [19], $H_0(\mathrm{GL}_N(\mathbb{Z}), \mathrm{St}_N) = 0$ when $N \geqslant 1$. Therefore we can compute $K_m(\mathbb{Z})$ if we understand the Hurewicz map

$$h_m : K_m(\mathbb{Z}) \to H_{m+1}(BQ, \mathbb{Z})$$

and if we compute the groups $H_{m+1-N}(\mathrm{GL}_N(\mathbb{Z}), \mathrm{St}_N)$ for all $N \leqslant m$.

9.1. **On the Hurewicz morphism.** Let $BQ = BQP(\mathbb{Z})$ be the classifying space of Quillen's Q-construction on the exact category $P(\mathbb{Z})$ of finitely generated free $\mathbb{Z}$-modules. By definition, for every integer $m \geqslant 1$,

$$K_{m-1}(\mathbb{Z}) = \pi_m(BQ)\,.$$

In this section we shall be interested in the kernel C_m of the Hurewicz map

$$h_m : \pi_m(BQ) \to H_m(BQ)\,,$$

where $H_m(X)$ stands for $H_m(X; \mathbb{Z})$.

Proposition 9.1. *The groups C_6 and C_7 lie in $\mathcal{S}_2$, and C_8 lies in $\mathcal{S}_5$.*

9.2. **Proof.** To prove this proposition, we use a morphism of spectra

$$K(\mathcal{E}) \to K(\mathbb{Z})$$

introduced by Rognes in [27], § 4, where $\mathcal{E}$ is the category of finite sets. At level zero this morphism is the map

$$\mathbb{Z} \times B\Sigma_\infty^+ \to \mathbb{Z} \times BGL(\mathbb{Z})^+ ,$$

where Σ_∞ is the infinite symmetric group, $GL(\mathbb{Z})$ is the infinite general linear group over $\mathbb{Z}$, and $(\ )^+$ is the +-construction of Quillen. Let ϕ be the fiber of that map and consider the fibration

$$\begin{tikzcd} B \arrow[r] & BQ \arrow[d] \\ & \phi \end{tikzcd} \tag{11}$$

where B is the first level of $K(\mathcal{E})$. When $m \geqslant 1$, the group

$$\pi_{m+1}(B) = \pi_m(\mathbb{Z} \times B\Sigma_\infty^+) = \pi_m^s$$

is the m-th homotopy group of spheres by the Barratt/Priddy/Quillen theorem. The map

$$\pi_m^s \to K_m(\mathbb{Z})$$

is an isomorphism modulo $\mathcal{S}_2$ when $m \leqslant 4$. Therefore the long exact sequence deduced from (11)

$$\cdots \to \pi_m^s \to K_m(\mathbb{Z}) \to \pi_{m+1}(\phi) \to \pi_{m-1}^s \to \cdots \tag{12}$$

implies that $\pi_m(\phi)$ lies in $\mathcal{S}_2$ when $m \leqslant 5$.

From [1] Theorem 1.5, which remains valid modulo a Serre class, it follows that the kernel of the Hurewicz map

$$\pi_m(\phi) \to H_m(\phi)$$

lies in $\mathcal{S}_2$ when $m = 6, 7$ or 8. On the other hand, π_5^s and π_6^s lie in $\mathcal{S}_2$, while π_7^s lies in $\mathcal{S}_5$. Using (12), this implies that the kernel of the map

$$K_{m-1}(\mathbb{Z}) \to \pi_m(\phi)$$

lies in $\mathcal{S}_2$ (resp $\mathcal{S}_5$) when $m = 6$ or 7 (resp. 8). The commutative diagram

$$\begin{tikzcd} K_{m-1}(\mathbb{Z}) \arrow[r] \arrow[d, "h_m"'] & \pi_m(\phi) \arrow[d] \\ H_m(BQ) \arrow[r] & H_m(\phi) \end{tikzcd}$$

concludes the proof.

Theorem 9.2. *We have $K_5(\mathbb{Z}) \cong \mathbb{Z}$ mod $\mathcal{S}_2$, $K_6(\mathbb{Z})$ lies in $\mathcal{S}_2$, and $K_7(\mathbb{Z})$ lies in $\mathcal{S}_5$.*

Proof.
- First we compute $K_5(\mathbb{Z})$. From Theorem 8.1, (i) and (ii), we know that, modulo $\mathcal{S}_2$, the group $H_{6-N}(GL_N(\mathbb{Z}), St_N)$ vanishes when $N \leqslant 5$ and $N \neq 3$, and that $H_3(GL_3(\mathbb{Z}), St_3) \cong \mathbb{Z}$.

  The exact sequence (10) for $N = 2$ reads

$$\begin{aligned} H_6(BQ_1, \mathbb{Z}) \ &\to \ H_6(BQ_2, \mathbb{Z}) \to H_4(GL_2(\mathbb{Z}), St_2) \to \\ &\to \ H_5(BQ_1, \mathbb{Z}) \to H_5(BQ_2, \mathbb{Z}) \to H_3(GL_2(\mathbb{Z}), St_2) \end{aligned}$$

Since $H_m(BQ_1,\mathbb{Z}) = 0$ when $m > 0$, $H_3(GL_2(\mathbb{Z}), St_2) \cong \hat{H}^{-3}(GL_2(\mathbb{Z}), St_2)$ is finite, and $H_4(GL_2(\mathbb{Z}), St_2)$ lies in $\mathcal{S}_2$, we conclude that $H_6(BQ_2,\mathbb{Z})$ lies in $\mathcal{S}_2$ and $H_5(BQ_2,\mathbb{Z})$ is finite.

The exact sequence (10) for $N = 3$ gives

$$H_6(BQ_2,\mathbb{Z}) \to H_6(BQ_3,\mathbb{Z}) \to H_3(GL_3(\mathbb{Z}), St_3) \to H_5(BQ_2,\mathbb{Z}),$$

therefore $H_6(BQ_3,\mathbb{Z}) \cong \mathbb{Z}$ modulo $\mathcal{S}_2$.

On the other hand, we deduce from (10) with $N = 5, 6, 7$ and from Theorem 8.1 that, modulo $\mathcal{S}_2$,

$$H_6(BQ,\mathbb{Z}) \cong H_6(BQ_5,\mathbb{Z}) \cong H_6(BQ_4,\mathbb{Z}),$$

as $H_1(GL_5(\mathbb{Z}), St_5)$ and $H_1(GL_6(\mathbb{Z}), St_6)$ are finite.

Now consider the exact sequence (10) for $N = 4$:

$$H_3(GL_4(\mathbb{Z}), St_4) \xrightarrow{\alpha} H_6(BQ_3,\mathbb{Z}) \to H_6(BQ_4,\mathbb{Z}) \to H_2(GL_4(\mathbb{Z}), St_4),$$

where the last group is in $\mathcal{S}_2$ by Theorem 8.1(ii).

If α were not zero modulo $\mathcal{S}_2$ then we would conclude that $H_6(BQ,\mathbb{Z}) \cong H_6(BQ_4,\mathbb{Z})$ is finite. But this is impossible since $K_5(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ (Borel) and the Hurewicz map

$$h_6 : K_5(\mathbb{Z}) \to H_6(BQ,\mathbb{Z})$$

has finite kernel.

Therefore $\alpha = 0$ modulo $\mathcal{S}_2$, and

$$H_6(BQ,\mathbb{Z}) \cong H_6(BQ_3,\mathbb{Z}) \cong \mathbb{Z}$$

modulo $\mathcal{S}_2$. The Hurewicz map h_6 is an isomorphism modulo torsion, and its kernel C_6 lies in $\mathcal{S}_2$ (Proposition 9.1) . Therefore $K_5(\mathbb{Z})$ is the direct sum of $\mathbb{Z}$ and a finite 2-group.

- Next, we compute $K_6(\mathbb{Z})$.
From Theorem 8.1(ii), we know that $H_{7-N}(GL_N(\mathbb{Z}), St_N)$ lies in $\mathcal{S}_2$ when $N \neq 4$, and, according to Theorem 8.1(i), we have $H_3(GL_4(\mathbb{Z}), St_4) \cong \mathbb{Z}$ modulo $\mathcal{S}_2$.

From the exact sequence (10) for $N = 2, 3$, we conclude from (9) that $H_7(BQ_3,\mathbb{Z})$ lies in $\mathcal{S}_2$. The exact sequence for $N = 5$ gives

$$\begin{aligned} H_7(BQ_3,\mathbb{Z}) &\to H_7(BQ_4,\mathbb{Z}) \to H_3(GL_4(\mathbb{Z}), St_4) \xrightarrow{\alpha} \\ &\xrightarrow{\alpha} H_6(BQ_3,\mathbb{Z}) \to H_6(BQ_4,\mathbb{Z}) \end{aligned}$$

Since $\alpha = 0$ modulo $\mathcal{S}_2$, we get $H_7(BQ_4,\mathbb{Z}) \cong H_3(GL_4(\mathbb{Z}), St_4) \cong \mathbb{Z}$ modulo $\mathcal{S}_2$. Using the exact sequence (10) for $N = 5$ and 6, we conclude that $H_7(BQ,\mathbb{Z}) \cong \mathbb{Z}$ modulo $\mathcal{S}_2$.

Since $K_6(\mathbb{Z})$ is finite (Borel) and the kernel C_7 of the Hurewicz map

$$h_7 : K_6(\mathbb{Z}) \longrightarrow H_7(BQ,\mathbb{Z})$$

lies in $\mathcal{S}_2$, we get that $K_6(\mathbb{Z})$ is a finite 2-group.

- Finally, we show that $K_7(\mathbb{Z})$ lies in $\mathcal{S}_5$. From Lemma 8.5 we deduce that the groups E^r_{pq} for $GL_N(\mathbb{Z})$, $N \leqslant 5$, $r \geqslant 1$, lie in $\mathcal{S}_5$ when $q > 0$.

Using Theorem 8.1 and Lemma 8.8, we conclude that $H_{8-N}(GL_N(\mathbb{Z}), St_N)$ lies in $\mathcal{S}_5$ when $N \leqslant 7$. This implies that $H_8(BQ,\mathbb{Z})$ is in $\mathcal{S}_5$ and, since the kernel C_8 of h_8 lies in $\mathcal{S}_5$ (Proposition 9.1), we conclude that $K_7(\mathbb{Z})$ has no p–torsion with $p > 5$.

□

***Remark* 9.3.** These three are already known: $K_5(\mathbb{Z}) \cong \mathbb{Z}$, $K_6(\mathbb{Z}) = 0$ and $K_7(\mathbb{Z}) \cong \mathbb{Z}/240$ (see [35]). The group $K_8(\mathbb{Z})$ still remains unknown.

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