J. K-Theory **12** (2013), 137–165
doi:10.1017/is013005031jkt229
©2013 ISOPP

Tame kernels and second regulators of number fields and their subfields

To Professor Aderemi O. Kuku
on the occasion of his 70th birthday

by

JERZY BROWKIN AND HERBERT GANGL*

Abstract

Assuming a version of the Lichtenbaum conjecture, we apply Brauer-Kuroda relations between the Dedekind zeta function of a number field and the zeta function of some of its subfields to prove formulas relating the order of the tame kernel of a number field F with the orders of the tame kernels of some of its subfields. The details are given for fields F which are Galois over $\mathbb{Q}$ with Galois group the group $\mathbb{Z}/2 \times \mathbb{Z}/2$, the dihedral group D_{2p}, p an odd prime, or the alternating group A_4. We include numerical results illustrating these formulas.

Key Words: Dedekind zeta functions, Bloch-Wigner dilogarithm, Bloch group, second regulator, Brauer-Kuroda relations.

Mathematics Subject Classification 2010: 11R42, 11G55.

1. Introduction

There are multiplicative relations between the Dedekind zeta functions of a number field and of some of its subfields, given by Brauer and Kuroda. The first nonvanishing coefficient of the Taylor expansion at $s = 0$ of the Dedekind zeta function of a number field is related to the class number and the first regulator of this field. Similarly, the analogous coefficient at $s = -1$ is related to the order of the tame kernel and the second regulator of the field (under the assumption of the Lichtenbaum conjecture).

We give more explicit statements in the case of a number field which is Galois over $\mathbb{Q}$ with Galois group the group $\mathbb{Z}/2 \times \mathbb{Z}/2$, or the dihedral group D_{2p}, p an odd prime, or the alternating group A_4.

*The first author thanks the organizers of the Conference in Nanjing University for the invitation and hospitality.

The paper is organized as follows. In Part I we recall known facts on Dedekind's zeta functions and on the leading coefficients of their Taylor expansions at $s = 0$ and $s = -1$. These coefficients depend on the class number, on the order of the tame kernel and on the corresponding regulators of the field in question.

In Part II we recall the Brauer-Kuroda relations and write them in an explicit form for several groups, including the groups mentioned above. We show that, for a biquadratic field, Brauer-Kuroda relations imply a well known expression of the class number of the field by means of class numbers of its quadratic subfields.

The main results of the paper (Theorems 1, 2 and 3, and Corollaries 1, 2 and 3) give relations between the second regulator, respectively the order of the tame kernel, of a field F with that of some of its subfields, where F is Galois over $\mathbb{Q}$ with Galois group $\mathbb{Z}/2\times\mathbb{Z}/2$, D_{2p}, or A_4. These results are proved under the assumption of Conjecture 1, which is a variant of the Lichtenbaum conjecture, combined with results of Bloch and Suslin.

In Part III we include results of numerical experiments, which give the (conjectural) values of the second regulator for some fields F of small degree over $\mathbb{Q}$, and give some evidence for Conjecture 1. We check its compatibility with the Brauer-Kuroda relations and give an example of two fields of different signature having the same second regulator numerically.

The first two parts of the paper contain an extended version of the talk given by the first author at the Conference in Nanjing University on the occasion of the 70th birthday of Professor Aderemi O. Kuku. The last part written by the second author presents numerical results giving some evidence for the conjecture mentioned above.

Part I. Dedekind zeta functions and their values at $s = 0$ and $s = -1$

2. The Dedekind zeta function

We recall the basic properties of the Dedekind zeta function $\zeta_F(s)$ of a number field F of a finite degree n over $\mathbb{Q}$.

It is a meromorphic function on $\mathbb{C}$ with a unique single pole at $s = 1$. It has zeros in the strip $\{s \in \mathbb{C} \ : \ 0 < \mathrm{Re}\, s < 1\}$ and possibly at nonpositive integers $-m$, $m \geq 0$. The multiplicity of zero at $s = -m$ equals

$$d_m = d_m(F) = \begin{cases} r_1 + r_2 - 1 & \text{if } m = 0, \\ r_1 + r_2 & \text{if } m \text{ is even, } m > 0, \\ r_2 & \text{if } m \text{ is odd.} \end{cases} \tag{2.1}$$

Here $r_1 = r_1(F)$ is the number of real places of F, and $r_2 = r_2(F)$ is the number of complex ones. We have $n = [F : \mathbb{Q}] = r_1 + 2r_2$.

The Dedekind zeta function satisfies a functional equation. To write it we need the following notation. Let

$$A(F) := \frac{|d(F)|^{1/2}}{2^{r_2}\pi^{n/2}},$$

where $d(F)$ is the discriminant of F, and let

$$\Phi(s) := A(F)^s \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \zeta_F(s).$$

Then the following functional equation holds: $\Phi(s) = \Phi(1-s)$ for $s \in \mathbb{C} \setminus \mathbb{Z}$. More explicitly,

$$A(F)^s \Gamma(\tfrac{s}{2})^{r_1} \Gamma(s)^{r_2} \zeta_F(s) = A(F)^{1-s} \Gamma(\tfrac{1-s}{2})^{r_1} \Gamma(1-s)^{r_2} \zeta_F(1-s). \tag{2.2}$$

Since $\Gamma(s)$ has poles at nonpositive integers and $\zeta_F(s)$ has a pole at $s = 1$, the formula (2.2) does not make sense for $s \in \mathbb{Z}$.

To overcome this difficulty we introduce the following notation. For an arbitrary function $f(s)$ whose Laurent expansion in a neighborhood of $s = s_0$ is

$$f(s) = a_r(s - s_0)^r + a_{r+1}(s - s_0)^{r+1} + \cdots, \quad \text{where} \quad r \in \mathbb{Z},\ a_r \neq 0, \tag{2.3}$$

we denote by $f^*(s_0)$ or by $(f(s_0))^*$ the first nonvanishing coefficient a_r in the expansion (2.3). Obviously $(f_1 f_2)^*(s_0) = f_1^*(s_0) \cdot f_2^*(s_0)$.

Then (2.2) implies that

$$A(F)^s \big(\Gamma(\tfrac{s}{2})^{r_1}\big)^* \big(\Gamma(s)^{r_2}\big)^* (\zeta_F(s))^* = A(F)^{1-s} \big(\Gamma(\tfrac{1-s}{2})^{r_1}\big)^* \big(\Gamma(1-s)^{r_2}\big)^* \big(\zeta_F(1-s)\big)^*, \tag{2.4}$$

and this formula holds for every $s \in \mathbb{C}$.

When substituting in (2.4) integer values for s, the following well known formula will be useful:

$$\Gamma^*(-n) = \frac{(-1)^n}{n!} \quad \text{for} \quad n \in \mathbb{Z},\ n \geq 0.$$

3. The value of $\zeta_F^*(0)$ and the first regulator

It is known (see [17], Theorem 7.3 and (6.8)) that

$$\zeta_F^*(1) = \frac{2^{r_1}(2\pi)^{r_2}}{|d(F)|^{1/2}} \cdot \frac{R_1(F)h(F)}{w_1(F)}, \tag{3.1}$$

where $w_1(F)$ is the number of roots of 1 in F, $h(F)$ is the class number of F, and $R_1(F)$ is the (first) regulator of F.

We recall here its definition (see e.g. [17] or [19]). Let $\mathcal{O}_F$ be the ring of algebraic integers in F, and let $\mathcal{O}_F^*$ be its group of units. The Dirichlet unit theorem says that $\mathcal{O}_F^*$ is the direct sum of the cyclic group of roots of unity in F of order $w_1(F)$, and a free abelian group of the rank $d_0 = r_1 + r_2 - 1$.

Let $\varepsilon_1, \ldots, \varepsilon_{d_0}$ be generators of this free abelian group. It is called a system of fundamental units of F. Let $\sigma_1, \sigma_2, \ldots, \sigma_{r_1+r_2}$ be embeddings $F \to \mathbb{C}$ corresponding to the archimedean places of F. The absolute value of the determinant

$$R_1(F) := \left|\det\bigl(c_i \log|\sigma_i(\varepsilon_j)|\bigr)_{1\le i,j\le d_0}\right|, \tag{3.2}$$

where $c_i = 1$ if σ_i is real, and $c_i = 2$ otherwise, does not depend on the choice of the fundamental units ε_j, and on the order of the places $\sigma_1, \sigma_2, \ldots, \sigma_{d_0}$ chosen. Since $d_0 = r_1 + r_2 - 1$, the archimedean place $\sigma_{r_1+r_2}$ has been omitted in (3.2).

We call $R_1(F)$ the first regulator of F.

From the functional equation (2.4) with $s = 0$ and (3.1) it follows that

$$\zeta_F^*(0) = -\frac{R_1(F)h(F)}{w_1(F)}. \tag{3.3}$$

In particular, if $d_0 = 0$, i.e. if $F = \mathbb{Q}$ or F is quadratic imaginary, then $R_1(F) = 1$. Hence (3.3) for $F = \mathbb{Q}$ gives

$$\zeta(0) = \zeta_{\mathbb{Q}}(0) = \zeta_{\mathbb{Q}}^*(0) = -\frac{1}{2},$$

and for $F = \mathbb{Q}(\sqrt{-d})$, d squarefree and > 0, we have

$$\zeta_F(0) = \zeta_F^*(0) = -\frac{h(F)}{w_1(F)},$$

where $w_1(F) = 4$ for $d = 1$, $w_1(F) = 6$ for $d = 3$, and $w_1(F) = 2$ otherwise.

For real quadratic fields F, we have $w_1(F) = 2$, and $R_1(F) = \log\varepsilon(F)$, where $\varepsilon(F) > 1$ is the fundamental unit of the field F. Thus (3.3) takes the form

$$\zeta_F^*(0) = -\tfrac{1}{2}\log\varepsilon(F)\cdot h(F).$$

4. The value of $\zeta_F^*(-1)$ and the second regulator

The results presented above concerning the case $s = 0$ are classical, and there are known effective algorithms for computing the values of the class number $h(F)$ and of the first regulator $R_1(F)$ of a number field F. See the computer algebra package [18], where these algorithms have been implemented. See also [11].

Our knowledge in the next cases $s = -1, -2, \ldots$ is less complete. In the present paper we do not discuss the cases $s \leq -2$. Instead we concentrate on the case $s = -1$.

By analogy with the formula (3.3) in the case $s = 0$, one can expect that an analogous formula holds in the case $s = -1$.

Namely, the first regulator $R_1(F)$ will be replaced by the second dilogarithmic regulator $\widetilde{R}_2(F)$ defined below in Section 6. It is the absolute value of the determinant of a matrix of size $d_1 = r_2(F)$.

The class number $h(F)$ of the field F, which is equal to the order of the torsion subgroup of the group K_0F, will be replaced by the order $k_2(F)$ of the tame kernel $K_2\mathcal{O}_F$ of F. See [16] for a definition.

Finally, the number $w_1(F)$ of roots of unity in F will be replaced by the number $w_2(F)$ of roots of unity in the compositum of all quadratic extensions of F.

Thus, by analogy with (3.3), (see also remarks before (12.1)) one can state the following conjecture :

Conjecture 1 *For every number field F we have*

$$|\zeta_F^*(-1)| = \frac{\widetilde{R}_2(F)k_2(F)}{w_2(F)}. \tag{4.1}$$

Let us remark that Conjecture 1 is related to the Birch-Tate and the Lichtenbaum conjectures, see [3]. For totally real fields F we have $r_2(F) = 0$, so $\widetilde{R}_2(F) = 1$ and (4.1) is the Birch-Tate conjecture.

5. The Bloch group

To define the second (or dilogarithmic) regulator, we need a definition of the Bloch group $\mathcal{B}(F)$ of a number field F (see [10]).

For any subfield E of $\mathbb{C}$ let $\mathbb{Z}[E]$ be the free abelian group with generators $[a]$, where a runs over all elements of E distinct from 0 and 1. Let $\partial_2 = \partial_2(E) : \mathbb{Z}[E] \to E^\times \widetilde{\wedge} E^\times$ be the homomorphism defined on the free generators by $\partial_2([a]) := a \widetilde{\wedge} (1-a)$. Here $\widetilde{\wedge}$ is a modified wedge product satisfying $u \widetilde{\wedge} (-u) = 0$ in place of the usual $u \wedge u = 0$.

Let $\mathcal{A}(E) := \ker \partial_2(E)$. Then we have the exact sequence

$$0 \to \mathcal{A}(E) \longrightarrow \mathbb{Z}[E] \xrightarrow{\partial_2(E)} E^\times \widetilde{\wedge} E^\times \xrightarrow{\nu} K_2E \to 0. \tag{5.1}$$

Here ν is defined by $\nu(a \widetilde{\wedge} b) = \{a, b\}$, where $\{a, b\} \in K_2F$ is the Steinberg symbol.

Let $\mathcal{C}(E)$ be the subgroup of $\mathbb{Z}[E]$ generated by the elements $[a]+[1-a]$, $[a]+[1/a]$, $a \neq 0,1$, and by the elements of the form $[a_1]+[a_2]+[a_3]+[a_4]+[a_5]$, (called 5-cycles), where $a_1,\dots,a_5 \in E^\times \setminus \{1\}$ satisfy $a_i a_{i+1} + a_{i+3} = 1$, for $i = 1,\dots,5$, and the indices are taken modulo 5. Hence $a_i a_{i+1} \neq 1$.

Obviously, every cyclic permutation of the elements in a 5-cycle gives the same 5-cycle. Moreover, the 5-cycle is determined by its first two arguments: If $a_1 = x$, $a_2 = y$, then $a_3 = \frac{1-x}{1-xy}$, $a_4 = 1-xy$ and $a_5 = \frac{1-y}{1-xy}$, since $xy = a_1a_2 \neq 1$.

One can easily verify that $\partial_2([a]+[1-a]) = \partial_2([a]+[1/a]) = 0$, and for every 5-cycle b we have $\partial_2(b) = 0$. Hence $\mathcal{C}(E) \subseteq \ker\partial_2 = \mathcal{A}(E)$. Defining the Bloch group of E by $\mathcal{B}(E) := \mathcal{A}(E)/\mathcal{C}(E)$ we get from (5.1) the exact sequence

$$0 \to \mathcal{B}(E) \longrightarrow \mathbb{Z}[E]/\mathcal{C}(E) \xrightarrow{\partial_2(E)} E^\times \widetilde{\wedge} E^\times \xrightarrow{\nu} K_2E \to 0.$$

6. The second regulator $\widetilde{R}_2(F)$

In the definition of the first regulator we considered the matrix of size d_0, with elements which are logarithms of some archimedean norms of fundamental units. In the case of the second regulator, we consider an analogous matrix of size d_1. The role of the units will be played by the elements of the Bloch group, and the logarithm will be replaced by the dilogarithm of Wigner and Bloch normalized as follows:

$$\widetilde{D}(z) := -\mathrm{Im}\left(\frac{1}{\pi}\int_1^z \frac{\log(1-t)}{t}dt\right) + \frac{\arg(1-z)}{\pi}\cdot\log|z|.$$

It differs by the factor $\frac{1}{\pi}$ from the original one $D(z)$ (see [3], Corollary 6.1.2).

It is a real analytic function $\widetilde{D}:\mathbb{C}\to\mathbb{R}$ satisfying $\widetilde{D}(\overline{z}) = -\widetilde{D}(z)$, where $\overline{z}$ is the complex conjugate of z. Hence $\widetilde{D}$ vanishes on $\mathbb{R}$.

The mapping $\widetilde{D}$ can be extended by linearity to a homomorphism $\mathbb{Z}[\mathbb{C}]\to\mathbb{R}$, defined on generators, by $\widetilde{D}([a]) := \widetilde{D}(a)$ for $a \in \mathbb{C}$. It can be proved that $\widetilde{D}(b) = 0$ for every element $b \in \mathcal{C}(E)$. Hence $\widetilde{D}$ induces a homomorphism

$$\widetilde{D} : \mathbb{Z}[E]/\mathcal{C}(E) \to \mathbb{R}, \quad \text{where} \quad E \subseteq \mathbb{C},$$

called also the dilogarithm.

Now let us return to the number field F. Let σ_j, $j = 1,2,\dots,r_2$ be the complex places of F.

Then $\widetilde{D}_j := \widetilde{D}\circ\sigma_j$ are homomorphisms $\mathbb{Z}[F]/\mathcal{C}(F)\to\mathbb{R}$ for $j = 1,2,\dots,r_2$. Collecting them we get a homomorphism $\mathbb{D} : \mathbb{Z}[F]/\mathcal{C}(F)\to\mathbb{R}^{r_2}$ defined by $\mathbb{D} := (\widetilde{D}_1,\dots,\widetilde{D}_{r_2})$. Since $\mathcal{B}(F)$ is a subgroup of $\mathbb{Z}[F]/\mathcal{C}(F)$, we can restrict $\mathbb{D}$ to this subgroup. It turns out that $\mathbb{D}(\mathcal{B}(F))$ is a lattice $\Lambda_2(F)$ of maximal rank in $\mathbb{R}^{r_2}$. We define the second dilogarithmic regulator $\widetilde{R}_2(F)$ as the covolume of this lattice.

In other words, if for some $b_1,\dots,b_{r_2} \in \mathcal{B}(F)$ the vectors $\mathbb{D}(b_1),\dots,\mathbb{D}(b_{r_2}) \in \mathbb{R}^{r_2}$ generate the lattice $\Lambda_2(F)$, then

$$\widetilde{R}_2(F) = |\det(\widetilde{D}(\sigma_j(b_i))_{1\le i,j\le r_2})|.$$

Part II. Brauer–Kuroda relations

7. Brauer–Kuroda relations

R. Brauer [5] and S. Kuroda [15] have independently given multiplicative relations between the zeta function of a number field and zeta functions of some of its subfields.

Let F/k be a Galois extension of number fields with the Galois group G. Then the following multiplicative relation holds.

For every cyclic subgroup H of G, let

$$c(H) := \frac{1}{(G:H)} \sum_{\substack{H^*\text{-cyclic} \\ H \subseteq H^* \subseteq G}} \mu\big(|H^*/H|\big),$$

where μ is the Möbius function.

Then, writing F^H for the fixed field of H in F, we have

$$\zeta_k(s) = \prod_{\substack{H\text{-cyclic} \\ H \subseteq G}} \zeta_{F^H}^{c(H)}(s). \tag{7.1}$$

In what follows we usually assume that $k = \mathbb{Q}$, so that $\zeta_k = \zeta$ is the Riemann zeta function.

Substituting $s = 0$ in (7.1), in view of (3.3), we get multiplicative relations between class numbers and the first regulators of corresponding fields. There are many papers devoted to this subject.

When we substitute $s = -1$, and apply Conjecture 1, we get conjectural relations between the orders of tame kernels and the second regulators of the fields in question.

We illustrate this by some simple examples. Let us observe that in fact the relation (7.1) depends essentially on the structure of the Galois group G of the field F only, and not on the field F itself.

Example 1 Let G be the cyclic group of order n. Then there exists a unique cyclic subgroup H of order d, for every $d\,|n$. The subgroups H^* containing H have orders dd', where $d'|n/d$. Consequently

$$\sum_{H^*}\mu(|H^*/H|) = \sum_{d'|n/d}\mu(d') = \begin{cases} 1, & \text{if} \quad n/d = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $c(H) = 1$ if $H = G$, and $c(H) = 0$, otherwise. From (7.1) we get

$$\zeta_k(s) = \zeta_{F^G}(s),$$

which is not interesting, since $F^G = k$.

Example 2 Let $G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle\sigma_1,\sigma_2\rangle$.

Then $H_0 = \langle\sigma_1\sigma_2\rangle$, $H_1 = \langle\sigma_1\rangle$, $H_2 = \langle\sigma_2\rangle$, and $E = \{1\}$ are all cyclic subgroups of G.

Since H_0, H_1, H_2 are maximal cyclic subgroups, we get

$$c(H_i) = \tfrac{1}{2}\mu(1) = \tfrac{1}{2} \qquad \text{for} \quad i = 0,1,2.$$

Next,

$$c(E) = \tfrac{1}{4}(3\mu(2) + \mu(1)) = -\tfrac{1}{2}.$$

For $i = 0,1,2$, let $F_i := F^{H_i}$. Then (7.1) gives

$$\zeta_k(s) = (\zeta_{F_0}(s)\zeta_{F_1}(s)\zeta_{F_2}(s))^{1/2}\zeta_F(s)^{-1/2}.$$

Hence

$$\zeta_F(s)\zeta_k(s)^2 = \zeta_{F_0}(s)\zeta_{F_1}(s)\zeta_{F_2}(s). \tag{7.2}$$

Example 3 Let $G = S_3$.

Let H_0 be the subgroup of G of order 3, and let H_1, H_2, H_3 be subgroups of order 2. They are conjugate. To these subgroups there correspond subfields of F : The quadratic subfield F_0, and the cubic ones F_1, F_2, F_3, which are isomorphic.

We have

$$\begin{aligned} c(E) &= \tfrac{1}{6}(3\mu(2) + \mu(3) + \mu(1)) = -\tfrac{1}{2}, \\ c(H_i) &= \tfrac{1}{3}\mu(1) = \tfrac{1}{3} \quad \text{for} \quad i = 1,2,3, \\ c(H_0) &= \tfrac{1}{2}\mu(1) = \tfrac{1}{2}. \end{aligned}$$

Then (7.1) gives

$$\zeta_F\zeta_k^2 = \zeta_{F_0}(\zeta_{F_1}\zeta_{F_2}\zeta_{F_3})^{2/3}.$$

Since zeta functions of isomorphic fields are equal, we get

$$\zeta_F\zeta_k^2 = \zeta_{F_0}\zeta_{F_1}^2. \tag{7.3}$$

In the following examples, we leave the details to the reader. Denote by ζ_σ, the zeta function of the subfield of F fixed by the automorphism σ.

Example 4 Let $G = A_4$.

The formula (7.1) gives

$$\zeta_F \zeta_k^2 = \zeta_{(12)(34)} \zeta_{(234)}^2.$$

Example 5 Let $G = S_4$.

Then

$$\zeta_F \zeta_k^2 = \zeta_{(12)} \zeta_{(1234)} \zeta_{(123)}.$$

Example 6 Let $G = S_5$.

Then

$$\zeta_F \zeta_k^4 = \zeta_{(123)(45)}^2 \zeta_{(1234)}^2 \zeta_{(12345)}.$$

Let us remark that other cyclic subgroups of G do not contribute to this formula.

Example 7 Let $G = D_{2p}$ be the dihedral group of order $2p$, where p is an odd prime. Let H_p be its unique subgroup of order p, and H_2 a subgroup of order 2. There are p subgroups of order 2 and they are conjugate.

Then

$$\zeta_F \zeta_k^2 = \zeta_{F^{H_p}} \zeta_{F^{H_2}}^2. \tag{7.4}$$

The case $p = 3$ has been treated above in Example 3 with more details, since $D_6 \cong S_3$.

Example 8 Let $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of quaternions. It has one cyclic subgroup $\langle -1 \rangle$ of order 2, and three cyclic subgroups $\langle i \rangle$, $\langle j \rangle$, $\langle k \rangle$ of order 4.

We have

$$\begin{aligned} c(E) &= \frac{1}{8}(\mu(1) + \mu(2) + 3\mu(4)) = 0, \\ c(\langle -1 \rangle) &= \frac{1}{4}(\mu(1) + 3\mu(2)) = -\frac{1}{2}, \\ c(\langle i \rangle) = c(\langle j \rangle) = c(\langle k \rangle) &= \frac{1}{2}\mu(1) = \frac{1}{2}. \end{aligned}$$

Here the Brauer-Kuroda relation takes the form

$$\zeta_{F^{\langle -1 \rangle}}(s) \zeta_k(s)^2 = \zeta_{F^{\langle i \rangle}}(s) \zeta_{F^{\langle j \rangle}}(s) \zeta_{F^{\langle k \rangle}}(s). \tag{7.5}$$

The Dedekind zeta function $\zeta_F(s)$ of the field F does not appear in this relation, because $c(E) = 0$.

Since $Q/\langle -1 \rangle = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $F^{\langle i \rangle}$, $F^{\langle j \rangle}$, $F^{\langle k \rangle}$ are quadratic subfields of $F^{\langle -1 \rangle}$, (7.5) is simply the relation (7.2) from Example 2 for the extension $F^{\langle -1 \rangle}/k$.

We call a finite group G exceptional, if the coefficient $c_G(E) = c(E) = 0$. Thus cyclic groups and the group of quaternions are exceptional. In [7] it has been proved that the only exceptional p-groups are cyclic and generalized quaternion. Consequently a nilpotent group is exceptional iff some of its Sylow subgroups are cyclic or generalized quaternion. In [7] there are given also many other examples of exceptional groups, e.g. $\mathrm{SL}(2,\mathbb{F}_q)$ with q odd is exceptional.

In all the above examples, except the first and the last, the Brauer-Kuroda relation is of the form $\zeta_F\,\zeta_k^m =$ a product of zeta functions of some proper subfields of F, with $m > 0$, because the value of $c(E)$ is negative, and $c(H) \geq 0$ for nontrivial cyclic subgroups H of G.

In general this is not the case. For example for $G = S_7$ and $\mathbb{Z}/6\times\mathbb{Z}/6$, we have $c(E) > 0$. Hence ζ_k and ζ_F are on different sides of the Brauer-Kuroda equation.

If the Galois group of a number field F is exceptional, then the Dedekind zeta function of the field does not appear in the Brauer-Kuroda relation (7.1). One may expect that then ζ_F is, in a sense, independent of ζ_k and of the Dedekind zeta functions of some other proper subfields of F.

This can be illustrated by the following observation. There are known Galois extensions F of $\mathbb{Q}$ with the quaternion Galois group such that $\zeta_F(\frac{1}{2}) = 0$ and $\zeta_{F_j}(\frac{1}{2}) \neq 0$ for proper subfields F_j of F.

Hence ζ_F is multiplicatively independent of ζ and ζ_{F_j}. (See [12]).

8. The case of a biquadratic field and $s = 0$

Let F be a biquadratic extension of $\mathbb{Q}$, and let F_0, F_1, F_2 be its quadratic subfields, with F_0 real. Then (7.2) holds.

First we substitute $s = 0$ in (7.2) and we get some well known relations between the class numbers and the first regulators of F and of its subfields.

Next we substitute $s = -1$ and assuming Conjecture 1, we discuss analogous relations between orders of the tame kernels and of the second regulators of the fields in question.

We are looking for some analogies in these two situations.

The most interesting case is when F is imaginary. Then F_1 and F_2 are quadratic imaginary, so their first regulators are trivial, $R_1(F_1) = R_1(F_2) = 1$. Moreover $r_2(F) = 2$ and $r_1(F_0) = 2$, so $d_0(F) = d_0(F_0) = 1$. Thus in F and in F_0, there is only one fundamental unit. Denote it by ε and by ε_0, respectively. Then $R_1(F_0) = \log|\varepsilon_0|$ and $R_1(F) = 2\log|\varepsilon|$. We may assume that $|\varepsilon| > 1$ and $|\varepsilon_0| > 1$.

Moreover, since ε_0 is a unit of F, we get $|\varepsilon_0| = |\varepsilon|^{Q_1(F)}$ for some $Q_1(F) \in \mathbb{N}$.

Consequently we obtain a regulator relation

$$R_1(F) = \frac{2}{Q_1(F)} R_1(F_0). \tag{8.1}$$

It is known that there are exactly two possibilities: $\varepsilon_0 = \varepsilon$ (then $Q_1(F) = 1$) and $\varepsilon_0 = \zeta\varepsilon^2$, where ζ is a root of unity (then $Q_1(F) = 2$). Some sufficient conditions for $Q_1(F) = 1$ are known. E.g.

- (i) If $N\varepsilon_0 = -1$, then $Q_1(F) = 1$.
- (ii) Let $F = \mathbb{Q}(\sqrt{-d_1}, \sqrt{-d_2})$, where d_1, d_2 are positive and squarefree. If $\gcd(d_1, d_2)$ has an odd prime factor, or both d_1, d_2 are even and $(d_1 d_2)/4 \equiv 1 \pmod 4$, then $Q_1(F) = 1$.

A more precise description of conditions equivalent to $Q_1(F) = 1$ is given in [14].

One can easily verify that for the biquadratic field F and its quadratic subfields F_0, F_1, F_2, we have

$$4w_1(F) = w_1(F_0)w_1(F_1)w_1(F_2)$$

with only one exception: $F = \mathbb{Q}(\zeta_4, \sqrt{2}) = \mathbb{Q}(\zeta_8)$.

Taking into account the value $\zeta(0) = -\frac{1}{2}$, the formula (7.2) for $s = 0$ gives

$$R_1(F)h(F) = R_1(F_0)h(F_0)h(F_1)h(F_2). \tag{8.2}$$

Hence for $F \neq \mathbb{Q}(\zeta_8)$, we get from (8.1) and (8.2) that

$$h(F) = \frac{Q_1(F)}{2} \prod_{j=0}^{2} h(F_j) \quad \text{with} \quad Q_1(F) = 1 \text{ or } 2. \tag{8.3}$$

If F is a totally real biquadratic field, then an analogous formula holds (see [14]):

$$h(F) = \frac{Q_1(F)}{4} \prod_{j=0}^{2} h(F_j) \quad \text{with} \quad Q_1(F) | 4.$$

9. The case of a biquadratic field and $s = -1$

We assume that F is a complex biquadratic extension of $\mathbb{Q}$, and we use the notation from the last section. Substituting $s = -1$ into (7.2) and assuming Conjecture 1, we get because $\zeta(-1) = -\frac{1}{12}$ that

$$\frac{\widetilde{R}_2(F)k_2(F)}{w_2(F)} \frac{1}{12^2} = \widetilde{R}_2(F_1)\widetilde{R}_2(F_2) \prod_{j=0}^{2} \frac{k_2(F_j)}{w_2(F_j)}. \tag{9.1}$$

One can easily verify that

$$w_2(F) = w_2(F_0) = \begin{cases} 2\cdot 24 & \text{if} \quad \sqrt{2} \in F, \\ 5\cdot 24 & \text{if} \quad \sqrt{5} \in F, \\ 24 & \text{otherwise,} \end{cases}$$

and $w_2(F_1) = w_2(F_2) = 24$. Hence (9.1) implies

$$\widetilde{R}_2(F)k_2(F) = \frac{1}{4}\widetilde{R}_2(F_1)\widetilde{R}_2(F_2)\prod_{j=0}^{2} k_2(F_j). \tag{9.2}$$

To proceed further we need a regulator formula for second regulators analogous to (8.1). We shall prove that $2\widetilde{R}_2(F_1)\cdot\widetilde{R}_2(F_2)$ is an integer multiple of $\widetilde{R}_2(F)$.

The lattices $\Lambda_2(F_1)$ and $\Lambda_2(F_2)$ corresponding to the Bloch groups $\mathcal{B}(F_1)$ and $\mathcal{B}(F_2)$ are 1-dimensional, since $r_2(F_1) = r_2(F_2) = 1$.

For $i = 1,2$ let $b_i \in \mathcal{B}(F_i)$ define a generator $\widetilde{D}(b_i)$ of the lattice $\Lambda_2(F_i)$. Hence $\widetilde{R}(F_i) = \widetilde{D}(b_i)$.

From $r_2(F) = 2$ it follows that the lattice $\Lambda_2(F)$ is 2-dimensional. Obviously $b_1, b_2 \in \mathcal{B}(F)$, hence $\mathbb{D}(b_1), \mathbb{D}(b_2) \in \Lambda_2(F)$.

Let $G = \mathrm{Gal}(F/\mathbb{Q})$. For $i = 1,2$, denote by $\tau_i \in G$ the nontrivial automorphism of F trivial on F_i. Then $\tau_1\tau_2$ is trivial on the real subfield F_0, hence it is complex conjugation. Consequently the two complex places of F are represented by $\sigma_1 = \mathrm{id}$ and $\sigma_2 = \tau_2$.

The sublattice Λ_2' generated by $\mathbb{D}(b_1)$ and $\mathbb{D}(b_2)$ in $\Lambda_2(F)$ has covolume equal to the absolute value of the determinant of the matrix

$$\begin{pmatrix} \mathbb{D}(b_1) \\ \mathbb{D}(b_2) \end{pmatrix} = \begin{pmatrix} \widetilde{D}(\sigma_1(b_1)) & \widetilde{D}(\sigma_2(b_1)) \\ \widetilde{D}(\sigma_1(b_2)) & \widetilde{D}(\sigma_2(b_2)) \end{pmatrix} = \begin{pmatrix} \widetilde{D}(b_1) & -\widetilde{D}(b_1) \\ \widetilde{D}(b_2) & \widetilde{D}(b_2) \end{pmatrix},$$

since $\sigma_2(b_1) = \tau_2(b_1) = \tau_2\tau_1(b_1)$ is the complex conjugate of b_1.

Thus

$$\mathrm{covol}(\Lambda_2') = 2\widetilde{D}(b_1)\widetilde{D}(b_2) = 2\widetilde{R}_2(F_1)\widetilde{R}_2(F_2).$$

The covolume of a sublattice is an integer multiple of the covolume of the lattice. Therefore $\mathrm{covol}(\Lambda_2') = Q_2(F)\mathrm{covol}(\Lambda_2(F))$ for some $Q_2(F) \in \mathbb{N}$. Thus we have proved

Theorem 1 *If F is a complex biquadratic extension of $\mathbb{Q}$ with imaginary quadratic subfields F_1 and F_2, then*

$$\widetilde{R}_2(F) = \frac{2}{Q_2(F)}\widetilde{R}_2(F_1)\widetilde{R}_2(F_2), \quad \textit{for some} \quad Q_2(F) \in \mathbb{N}. \tag{9.3}$$

From (9.2) and (9.3) we get

Corollary 1 *Assume Conjecture* 1 *for the fields in question. Then in the notation of Theorem* 1, *we have*

$$k_2(F) = \frac{Q_2(F)}{8} \prod_{j=0}^{2} k_2(F_j) \qquad \textit{for some} \quad Q_2(F) \in \mathbb{N}. \tag{9.4}$$

Zhou Haiyan [23] proved that the odd parts of the k_2's on both sides of (9.4) are equal. It follows that $Q_2(F)$ is [1] a power of 2.

On the basis of numerical evidence given below in Section 12.3, we expect that $Q_2(F)$ is always 1 or 2.

If F is a real biquadratic field not containing $\sqrt{2}$ nor $\sqrt{5}$, and if F_0, F_1, F_2 are its quadratic subfields, then w_2 of all these fields equals 24, and their second regulators are equal 1.

Applying the Birch–Tate conjecture, we get

$$k_2(E) = w_2(E)|\zeta_E(-1)|,$$

which has been proved already for all totally real abelian fields E (see [22] Theorem 1.5 and note on p.499). From the Brauer–Kuroda relation (7.2) and the fact that $\zeta(-1) = -\frac{1}{12}$, we get

$$k_2(F) = \tfrac{1}{4} \prod_{j=0}^{2} k_2(F_j). \tag{9.5}$$

Now, assuming Conjecture 1, we give a numerical example, which indicates that the regulator index $Q_2(F)$ in (9.3) can be even, and thus can be greater than 1.

Example 9 (cf. [6]). Let $F = \mathbb{Q}(\sqrt{-6}, \sqrt{-15})$. Then $F_0 = \mathbb{Q}(\sqrt{10})$, $F_1 = \mathbb{Q}(\sqrt{-15})$, $F_2 = \mathbb{Q}(\sqrt{-6})$ are quadratic subfields of F. For all these fields, w_2 equals 24 and $\widetilde{R}_2(F_0) = 1$, since F_0 is real.

1) The number $a = \frac{1+\sqrt{-15}}{4} \in F_1$ satisfies $a^2 - \frac{1}{2}a + 1 = 0$, and hence $1 - a^3 = -(1-a)^3$. Taking $b_1 := 18[a] - 2[a^3]$ for $\partial_{21} := \partial_2(F_1)$ we get

$$\begin{aligned} \partial_{21}(b_1) &= 18(a \widetilde{\wedge} (1-a)) - 2(a^3 \widetilde{\wedge} (1-a^3)) \\ &= 18(a \widetilde{\wedge} (1-a)) - (a^3 \widetilde{\wedge} (1-a)^6) = 0. \end{aligned}$$

Hence $b_1 \in \mathcal{A}(F_1)$ and $\widetilde{D}(b_1) \in \Lambda_2(F_1)$.

[1] Note added on January 27, 2013: She proved recently that $Q_2(F) = 1, 2$ or 4 , see [24].

Assuming Conjecture 1 for the field F_1, we get

$$|\zeta^*_{F_1}(-1)| = \frac{\widetilde{R}_2(F_1)k_2(F_1)}{w_2(F_1)}. \tag{9.6}$$

Since $k_2(F_1) = 2$, $w_2(F_1) = 24$, $\zeta^*_{F_1}(-1) = -0.499525$ and $\widetilde{D}(b_1) = 5.99431$, it follows from (9.6) that $\widetilde{R}_2(F_1) = \widetilde{D}(b_1)$, i.e. $\widetilde{D}(b_1)$ generates the lattice $\Lambda_2(F_1)$.

2) We have $a = a_1^2$, where $a_1 = \frac{\sqrt{10}+\sqrt{-6}}{4} \in F$. Therefore in $F^\times \widetilde{\wedge} F^\times$ we have

$$a^3 \widetilde{\wedge}(-(1-a)^3) = a_1^6 \widetilde{\wedge}(-(1-a)^3) = a_1^6 \widetilde{\wedge}(1-a)^3 = a^3 \widetilde{\wedge}(1-a)^3.$$

Consequently, for $\partial_2 := \partial_2(F)$ we get

$$\partial_2(b_1/2) = 9(a \widetilde{\wedge}(1-a)) - (a^3 \widetilde{\wedge}(1-a^3)) = 9(a \widetilde{\wedge}(1-a)) - (a^3 \widetilde{\wedge}(1-a)^3) = 0.$$

Hence $b_1/2 \in \mathcal{A}(F)$ and $\widetilde{D}(b_1/2) = \frac{1}{2}\widetilde{D}(b_1) \in \Lambda_2(F)$.

Let $\widetilde{D}(b_2)$ be a generator of the lattice $\Lambda_2(F_2)$, where $b_2 \in \mathcal{A}_2$. Then $\mathbb{D}(b_1/2)$ and $\mathbb{D}(b_2)$ generate a sublattice of $\Lambda_2(F)$ of covolume

$$\left|\det\begin{pmatrix}\mathbb{D}(b_1/2)\\ \mathbb{D}(b_2)\end{pmatrix}\right| = \frac{1}{2}\left|\det\begin{pmatrix}\mathbb{D}(b_1)\\ \mathbb{D}(b_2)\end{pmatrix}\right| = \widetilde{R}_2(F_1)\widetilde{R}_2(F_2).$$

Therefore $\widetilde{R}_2(F_1)\widetilde{R}_2(F_2)$ is an integer multiple of $\widetilde{R}_2(F)$ and from (9.3) it follows that $Q_2(F)$ is even.

10. The case of the dihedral Galois group D_{2p} and $s = -1$

Let D_{2p} be the dihedral group of order $2p$, where p is an odd prime. It is the group of isometries of a regular p-gon with p vertices $1, 2, \ldots, p$.

The group has a unique subgroup of order p generated by the rotation $\tau = (123\ldots p)$, and p subgroups of order 2 generated by symmetries.

Let $\sigma = \sigma_p$ be the symmetry fixing the vertex p,

$$\sigma = (1, p-1)(2, p-2)\cdots(\tfrac{p-1}{2}, \tfrac{p+1}{2}).$$

Other symmetries are $\sigma_j := \tau^j \sigma \tau^{-j}$, $j = 1, 2, \ldots, p-1$. Then σ_j fixes the vertex j.

Let F be a Galois extension of $\mathbb{Q}$ with the Galois group $G = D_{2p}$. It has a unique quadratic subfield F_0 fixed by τ, and p subfields F_j fixed by σ_j, $j = 1, 2, \ldots, p$, of degree p.

We have $\tau(F_j) = F_{j+1}$ for $j = 1, 2, \ldots, p-1$ and $\tau(F_p) = F_1$.

Namely, if $a \in F_j$ and $\sigma_j(a) = a$ then $\tau(a)$ satisfies

$$\sigma_{j+1}(\tau(a)) = \tau^{j+1}\sigma\tau^{-j}(a) = \tau(\sigma_j(a)) = \tau(a).$$

Therefore $\tau(a) \in F_{j+1}$ and $F_j = \tau^j(F_p)$ for $j = 1,2,\ldots,p$.

Assume that the field F is complex. Then complex conjugation belongs to G, we may assume that $\sigma = \sigma_p$ is the complex conjugation.

Then the field F_p fixed by σ is the unique maximal real subfield of F.

We have

$$r_2(F_0) = 1, \quad r_2(F_j) = \tfrac{p-1}{2}, \quad j = 1,2,\ldots,p, \quad r_2(F) = p.$$

We determine the complex places of the fields F_0, F_1, F_p and F.

Obviously, id is the complex place of F_0, and τ^j, $j = 0,1,\ldots,p-1$ are complex places of F.

Since $\sigma\tau^j$ is a symmetry, we get $\sigma\tau^j\sigma\tau^j$ =id, hence $\sigma\tau^j = \tau^{-j}\sigma$. Consequently

$$\sigma(F_j) = \sigma\tau^j(F_p) = \tau^{-j}\sigma(F_p) = \tau^{p-j}(F_p) = F_{p-j}$$

for $j = 1,2,\ldots,p-1$. It follows that the fields F_j and F_{p-j} are complex conjugate. Therefore the complex places of F_p are

$$\tau, \tau^2, \ldots, \tau^t, \quad \text{where} \quad t = \tfrac{p-1}{2},$$

and the complex places of $F_1 = \tau(F_p)$ are

$$\mathrm{id}, \tau, \ \tau^2, \ldots, \tau^{t-1}.$$

Now we describe the dilogarithmic lattices of the fields F_0, F_p and F_1.

The dilogarithmic lattice $\Lambda_2(F_0)$ of rank 1 is generated by $\widetilde{D}(b_0)$ for some $b_0 \in \mathcal{B}(F_0)$. Hence

$$\widetilde{R}_2(F_0) = \widetilde{D}(b_0). \tag{10.1}$$

The dilogarithmic lattice $\Lambda_2(F_p)$ of rank t is generated by the following vectors $\mathbb{D}_{F_p}(b_1)$,..., $\mathbb{D}_{F_p}(b_t)$ for some $b_1,\ldots,b_t \in \mathcal{B}(F_p)$, where

$$\mathbb{D}_{F_p}(b_j) = (\widetilde{D}(\tau(b_j)), \widetilde{D}(\tau^2(b_j)), \ldots, \widetilde{D}(\tau^t(b_j)).$$

Consequently,

$$\widetilde{R}_2(F_p) = |\det(U_1, U_2, \ldots, U_t)|, \quad \text{where} \quad U_j = \begin{pmatrix} \widetilde{D}(\tau^j(b_1)) \\ \cdots \\ \widetilde{D}(\tau^j(b_t)) \end{pmatrix}, \quad j = 1,2,\ldots,t. \tag{10.2}$$

Similarly, the dilogarithmic lattice $\Lambda_2(F_1)$ of rank t is generated by the following vectors $\mathbb{D}_{F_1}(b_{t+1}),\ldots,\mathbb{D}_{F_1}(b_{2t})$ for some $b_{t+1},\ldots,b_{2t} \in \mathcal{B}(F_1)$, where

$$\mathbb{D}_{F_1}(b_j) = (\widetilde{D}(b_j),\widetilde{D}(\tau(b_j)),\ldots,\widetilde{D}(\tau^{t-1}(b_j))).$$

Consequently,

$$\widetilde{R}_2(F_1) = |\det(V_1,V_2,\ldots,V_t)|, \quad \text{where} \quad V_j = \begin{pmatrix} \widetilde{D}(\tau^{t+j}(b_{t+1})) \\ \cdots \\ \widetilde{D}(\tau^{t+j}(b_{2t})) \end{pmatrix}, \quad j = 1,2,\ldots,t. \tag{10.3}$$

Since the Bloch groups $\mathcal{B}(F_0),\mathcal{B}(F_p),\mathcal{B}(F_1)$ can be mapped canonically into $\mathcal{B}(F)$, the elements $b_0,b_1,\ldots,b_{2t}$ defined above can be considered as elements of $\mathcal{B}(F)$.

Therefore the lattice Λ_2' generated by elements $\mathbb{D}_F(b_j)$, $j = 0,1,\ldots,2t$, where $\mathbb{D}_F(b) = (\widetilde{D}(b),\widetilde{D}(\tau(b)),\widetilde{D}(\tau^2(b)),\ldots,\widetilde{D}(\tau^{p-1}(b)))$ for $b \in \mathcal{B}(F)$, is a sublattice of the dilogarithmic lattice $\Lambda_2(F)$.

We determine the covolume of Λ_2'. By the definition of Λ_2' we have

$$\operatorname{covol}(\Lambda_2') = |\det \begin{pmatrix} \mathbb{D}_F(b_0) \\ \mathbb{D}_F(b_1) \\ \cdots \\ \mathbb{D}_F(b_{2t}) \end{pmatrix}|. \tag{10.4}$$

The first row of this matrix is simply

$$(\widetilde{D}(b_0),\widetilde{D}(b_0),\ldots,\widetilde{D}(b_0)) = \widetilde{D}(b_0)(1,1,\ldots,1).$$

The $(j+1)$st row, where $1 \le j \le t$, is

$$(\widetilde{D}(\tau(b_j)),\widetilde{D}(\tau^2(b_j)),\ldots,\widetilde{D}(\tau^t(b_j)),\widetilde{D}(\tau^{t+1}(b_j)),\ldots,\widetilde{D}(\tau^{2t}(b_j)),\widetilde{D}(\tau^{2t+1}(b_j)))$$

$$= (\widetilde{D}(\tau(b_j)),\widetilde{D}(\tau^2(b_j)),\ldots,\widetilde{D}(\tau^t(b_j)),-\widetilde{D}(\tau^t(b_j)),\ldots,-\widetilde{D}(\tau(b_j)),0),$$

since $\tau^k(b_j)$ and $\tau^{p-k}(b_j)$ are complex conjugate and $\tau^{2t+1}(b_j) = \tau^p(b_j) = b_j$ is real.

The $(j+1)$st row, where $t+1 \le j \le 2t$, is

$$(\widetilde{D}(\tau(b_j)),\widetilde{D}(\tau^2(b_j)),\ldots,\widetilde{D}(\tau^{t-1}(b_j)),\widetilde{D}(\tau^t(b_j)),\widetilde{D}(\tau^{t+1}(b_j)),\ldots,\widetilde{D}(\tau^{2t}(b_j)),\widetilde{D}(\tau^{2t+1}(b_j)))$$

$$= (\widetilde{D}(\tau(b_j)),\widetilde{D}(\tau^2(b_j)),\ldots,\widetilde{D}(\tau^{t-1}(b_j)),-\widetilde{D}(\tau^{t-1}(b_j)),\ldots,-\widetilde{D}(\tau(b_j)),-\widetilde{D}(b_j),0,\widetilde{D}(b_j)),$$

since $\tau^k(b_j)$ and $\tau^{2t-k}(b_j)$ are complex conjugate and $\tau^{2t}(b_j) = \tau^{p-1}(b_j)$ is real.

Hence covol(Λ_2') is the absolute value of the determinant of the matrix

$$\begin{aligned}
&\widetilde{D}(b_0)\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ U_1 & U_2 & \cdots & U_{p-1} & U_p \\ V_2 & V_3 & \cdots & V_p & V_1 \end{pmatrix} \\
&= \widetilde{D}(b_0)\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ U_1 & U_2 & \cdots & U_{t-1} & U_t & -U_t & \cdots & -U_2 & -U_1 & 0 \\ V_2 & V_3 & \cdots & V_t & -V_t & -V_{t-1} & \cdots & -V_1 & 0 & V_1 \end{pmatrix},
\end{aligned} \tag{10.6}$$

where U_j, V_j are given by (10.2) and (10.3).

Lemma 1 (On the "circulant" matrix) *Let M be the last matrix in* (10.6), *where U_j and V_j are arbitrary column vectors of height t. Then*

$$|\det(M)| = (2t+1)|\det(U_1,\ldots,U_t)\cdot\det(V_1,\ldots,V_t)|.$$

Proof: We operate on columns of M as follows:

1) Add the column containing U_j to the column containing $-U_j$ for $j = 1,\ldots,t$. We obtain

$$\begin{pmatrix} \text{first } t \text{ columns} & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 1 \\ \text{as in } M & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & -V_t - V_{t-1} & V_t - V_{t-2} & V_{t-1} - V_{t-3} & \cdots & V_4 - V_2 & V_3 - V_1 & V_2 & V_1 \end{pmatrix}.$$

2) Add the jth column to the $(j-2)$nd consecutively for $j = 2t+1, 2t, 2t-1, \ldots, t+3$, i.e. we begin with the last column. We get

$$\begin{pmatrix} \text{first } t \text{ columns} & t+1 & t & t-1 & \cdots & 4 & 3 & 2 & 1 \\ \text{as in } M & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & -V_t & V_t & V_{t-1} & \cdots & V_4 & V_3 & V_2 & V_1 \end{pmatrix}.$$

3) Adding the $(t+2)$nd column to the $(t+1)$st, we get

$$M' := \begin{pmatrix} \text{first } t \text{ columns} & 2t+1 & t & t-1 & \cdots & 4 & 3 & 2 & 1 \\ \text{as in } M & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & 0 & V_t & V_{t-1} & \cdots & V_4 & V_3 & V_2 & V_1 \end{pmatrix}.$$

From the above it follows that $\det M = \det M'$. Now we apply to $\det M'$ the Laplace formula with respect to the $(t+1)$st column. We get

$$|\det M'| = (2t+1)\left|\det\begin{pmatrix} U_1 & U_2 & \cdots & U_t & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & * & V_t & V_{t-1} & \cdots & V_2 & V_1 \end{pmatrix}\right|.$$

Hence the lemma follows. □

Consequently, by (10.4), (10.5) and Lemma 1, we get

$$\operatorname{covol}(\Lambda_2') = p\widetilde{R}_2(F_0)\widetilde{R}_2(F_p)\widetilde{R}_2(F_1) = p\widetilde{R}_2(F_0)\widetilde{R}_2(F_1)^2, \tag{10.5}$$

since the isomorphic fields F_1 and F_p have equal second regulators, and $2t+1=p$.

Thus we have proved

Theorem 2 *If F is a Galois extension of $\mathbb{Q}$ with the dihedral Galois group D_{2p}, where p is an odd prime, and if F is not totally real and satisfies $w_2(F)=24$, then*

$$\widetilde{R}_2(F) = \frac{p}{Q_2(F)}\widetilde{R}_2(F_0)\widetilde{R}_2(F_1)^2 \quad \text{for some} \quad Q_2(F)\in\mathbb{N},$$

where F_0 is the unique quadratic subfield of F and F_1 is a subfield of degree p.

Corollary 2 *In the notation of Theorem 2, assume that Conjecture 1 holds for all fields in question and that they satisfy $w_2(\cdot)=24$. Then*

$$k_2(F) = \frac{Q_2(F)}{4p}k_2(F_0)k_2(F_1)^2 \quad \text{for some}^{2} \quad Q_2(F)\in\mathbb{N}.$$

11. The case of the alternating Galois group A_4 and $s=-1$

Let F be a complex field Galois over $\mathbb{Q}$ with the Galois group $G=A_4$. We can assume that $\sigma := (12)(34)$ is complex conjugation. Then $F_6 := F^{\sigma}$ is the maximal real subfield of F of degree 6 over $\mathbb{Q}$ and $F_4 := F^{(234)}$, $F_4' := F^{(124)}$ are isomorphic but not complex conjugate subfields of F of degree 4 over $\mathbb{Q}$. Let $\tau := (13)(24)$. Then $F_3 := F^{\langle\sigma,\tau\rangle}$ is a totally real cubic cyclic subfield of F_6.

We have $r_2(F_6)=r_2(F_4)=r_2(F_4')=2$ and $r_2(F)=6$. Let $\rho := (123)$. One can verify that the complex places for these fields are

ρ and ρ^{-1} for F_6,

id and τ for F_4 and F_4',

$\mathrm{id},\tau,\rho,(142),\rho^{-1}$ and (234) for F.

Let $b_1,b_2\in\mathcal{B}(F_6)$, $b_3,b_4\in\mathcal{B}(F_4)$, and $b_5,b_6\in\mathcal{B}(F_4')$ define the lattices $\Lambda_2(F_6)$, $\Lambda_2(F_4)$, and $\Lambda_2(F_4')$, respectively. Then

$$\widetilde{R}_2(F_6) = |\det\begin{pmatrix}\mathbb{D}(b_1)\\ \mathbb{D}(b_2)\end{pmatrix}|, \widetilde{R}_2(F_4) = |\det\begin{pmatrix}\mathbb{D}(b_3)\\ \mathbb{D}(b_4)\end{pmatrix}|, \text{ and } \widetilde{R}_2(F_4') = |\det\begin{pmatrix}\mathbb{D}(b_5)\\ \mathbb{D}(b_6)\end{pmatrix}|.$$

[2] Note added on January 27, 2013: Zhou Haiyan proved recently that in the case $p=3$ we have $Q_2(F)=1,3,9$ or 27, see [24].

Since the elements $b_j,\ j=1,\ldots,6$, belong to $\mathcal{B}(F)$, they define a sublattice $\Lambda_2'(F)$ of the lattice $\Lambda_2(F)$. We have

$$\mathrm{covol}(\Lambda_2'(F)) = |\det\begin{pmatrix} 0 & 0 & \widetilde{D}(b_3) & \widetilde{D}(b_4) & \widetilde{D}(b_5) & \widetilde{D}(b_6) \\ 0 & 0 & \widetilde{D}(\tau b_3) & \widetilde{D}(\tau b_4) & \widetilde{D}(\tau b_5) & \widetilde{D}(\tau b_6) \\ \widetilde{D}(\rho b_1) & \widetilde{D}(\rho b_2) & -\widetilde{D}(b_3) & -\widetilde{D}(b_4) & \widetilde{D}(\tau b_5) & \widetilde{D}(\tau b_6) \\ -\widetilde{D}(\rho b_1) & -\widetilde{D}(\rho b_2) & -\widetilde{D}(\tau b_3) & -\widetilde{D}(\tau b_4) & \widetilde{D}(b_5) & \widetilde{D}(b_6) \\ -\widetilde{D}(\rho^{-1} b_1) & -\widetilde{D}(\rho^{-1} b_2) & -\widetilde{D}(\tau b_3) & -\widetilde{D}(\tau b_4) & \widetilde{D}(\tau b_5) & \widetilde{D}(\tau b_6) \\ \widetilde{D}(\rho^{-1} b_1) & \widetilde{D}(\rho^{-1} b_2) & \widetilde{D}(b_3) & \widetilde{D}(b_4) & -\widetilde{D}(b_5) & -\widetilde{D}(b_6) \end{pmatrix}|.$$

After computing this determinant we get

$$\mathrm{covol}(\Lambda_2'(F)) = 4\widetilde{R}_2(F_6)\widetilde{R}_2(F_4)\widetilde{R}_2(F_4') = 4\widetilde{R}_2(F_6)\widetilde{R}_2(F_4)^2, \qquad (11.1)$$

since isomorphic fields have equal second regulators.

Let $Q_2(F)$ be the index of $\Lambda_2'(F)$ in $\Lambda_2(F)$. Then

$$\mathrm{covol}(\Lambda_2'(F)) = Q_2(F)\mathrm{covol}(\Lambda_2(F)) = Q_2(F)\widetilde{R}_2(F).$$

Thus, by (11.1), we have proved

Theorem 3 *If F is a complex Galois extension of $\mathbb{Q}$ with Galois group A_4, satisfying $w_2(F)=24$, then in the above notation we have*

$$\widetilde{R}_2(F) = \frac{4}{Q_2(F)}\widetilde{R}_2(F_6)\widetilde{R}_2(F_4)^2, \quad \textit{for some} \quad Q_2(F)\in\mathbb{N}.$$

Corollary 3 *Assuming Conjecture* 1 *for the fields in question and that $w_2(F)=24$, we have in the notation of Theorem* 3 *that*

$$(i) \qquad k_2(F) = \frac{Q_2(F)}{16}k_2(F_6)k_2(F_4)^2 \quad \textit{for some} \quad Q_2(F)\in\mathbb{N}.$$

and

$$(ii) \qquad k_2(F_6) = \frac{\widetilde{R}_2(F_4)}{2\widetilde{R}_2(F_6)}k_2(F_3)k_2(F_4).$$

Proof: From the Brauer-Kuroda relation given in Example 4, we get

$$\zeta_F\zeta^2 = \zeta_{F_6}\zeta_{F_4}^2. \qquad (11.2)$$

Then, by Conjecture 1, we obtain

$$4\widetilde{R}_2(F)k_2(F) = \widetilde{R}_2(F_6)\widetilde{R}_2(F_4)^2k_2(F_6)k_2(F_4)^2$$

and by Theorem 3, we get the first part of the corollary.

From the Brauer-Kuroda relation given in Example 2 applied to the Galois extension F/F_3 with Galois group $\mathbb{Z}/2\times\mathbb{Z}/2$, we get

$$\zeta_F\zeta_{F_3}^2=\zeta_{F_6}^3, \tag{11.3}$$

since all three sextic subfields of F are isomorphic.

Eliminating ζ_F from (11.2) and (11.3), we obtain

$$\zeta_{F_6}^2\zeta^2=\zeta_{F_3}^2\zeta_{F_4}^2.$$

Hence $\zeta_{F_6}\zeta=\zeta_{F_3}\zeta_{F_4}$, since for real $s>1$ every Dedekind zeta function of a number field takes positive values.

Then, by Conjecture 1, we get

$$2\widetilde{R}_2(F_6)k_2(F_6)=\widetilde{R}_2(F_4)k_2(F_3)k_2(F_4), \tag{11.4}$$

since $\widetilde{R}_2(F_3)=1$. Thus the regulators $\widetilde{R}_2(F_6)$ and $\widetilde{R}_2(F_4)$ differ by a rational factor. From (11.4) we obtain the second part of the corollary. □

Part III. Numerical Examples

12.1. Introduction

For a number of Galois extensions F of $\mathbb{Q}$ with dihedral Galois groups (i.e. of type D_{2p}, for some $p\geq 3$ not necessarily a prime), we compare the Brauer-Kuroda relation at $s=-1$ with the associated numerical regulator values.

We also consider biquadratic extensions and A_4-extensions of $\mathbb{Q}$, and find in the latter case a surprising coincidence of regulators, which result from *different* lattices of certain subfields.

Our set-up is the following: We assume that (a version of) the Lichtenbaum Conjecture in weight 2 holds and combine it with results of Bloch and Suslin which relate $K_3(F)$ to the Bloch group and the Borel regulator to the Bloch-Wigner dilogarithm function $D(z)$ given above.

Numerous experiments support and suggest the following formulation (cf. (4.1)):

$$\zeta_F^*(-1)\stackrel{?}{=}\pm\frac{k_2(F)\widetilde{R}_2(F)}{w_2(F)}, \tag{12.1}$$

where we put

- $k_2(F)=\#K_2\mathcal{O}_F$, the order of the K-group of the number ring, as in the text above,

- $w_2(F)$ also as in the text above,
- $\widetilde{R}_2(F) = \mathrm{covol}(\Lambda_2(F))$, where the lattice

$$\Lambda_2(F) = \left\langle (\frac{1}{\pi} D \circ \sigma_j(\xi))_j \,|\, \xi \in \ker(\partial_2 : \mathbb{Z}[F] \to F^\times \widetilde{\wedge} F^\times) \right\rangle$$

is generated by the images, under the normalized Bloch-Wigner dilogarithm function $\widetilde{D}(z) = \frac{1}{\pi} D(z)$, of all the elements in the Bloch group $\mathcal{B}(F)$, where the $\sigma_j : F \to \mathbb{C}$ represent r_2 complex embeddings of F (one for each pair of complex conjugate ones).

12.2. What we compute

Our program, written with the computer algebra package GP-PARI [18], finds a set of elements in the Bloch group, and in many cases sufficiently many of them to generate a sublattice of full rank of the lattice $\Lambda_2(F)$ and hence we get a meaningful covolume. With these data we can form the quotient of covolumes of the regulator lattices for the subfields of a given field and compare it to the theoretical prediction in the text above. Moreover, we can compare it to the corresponding Dedekind zeta values at $s = -1$, which we can conveniently obtain via Magma [4], and get conjectural values for the K_2-orders of the number rings involved. For more details see e.g. [9].

Caveat: We will use the notation $\doteq$ below to indicate that two sides are equal up to several digits (usually we work with a minimum precision of 30 digits), but often *under the further assumption* that we have found the actual lattice generated by the Bloch group. Note that we cannot prove in any single case that the elements found will suffice to generate the full group. In fact for fields of degree > 10, say, this assumption is presumably overly optimistic.

12.3. Biquadratic cases

The Galois group of a biquadratic extension F_4 of signature (0,2) is the Klein 4-group (note that this can be viewed as a dihedral group with 4 elements, i.e. as D_{2p} with $p = 2$, but its behaviour is rather different from the other D_{2p}, $p > 2$).

In a nutshell, the experiments hint at a strong correlation: if we denote the two imaginary quadratic subfields of F_4 by F_1 and F_2, then we find for the regulator quotients (see Theorem 1)

$$\frac{2}{Q_2(F)} = \frac{\widetilde{R}_2(F_4)}{\widetilde{R}_2(F_1)\widetilde{R}_2(F_2)} \doteq 1 \quad \text{or} \quad \doteq 2.$$

Moreover, the pattern evolving is that this quotient seems to be equal to 1 if the "Birch-Tate-Lichtenbaum quotients"

$$\frac{w_2(F)\zeta_F^*(-1)}{\widetilde{R}_2(F)}$$

for $F = F_1$ and $F = F_2$ (which are conjecturally equal to the orders of the K-groups $\#K_2(\mathcal{O}_{F_1})$ and $\#K_2(\mathcal{O}_{F_2})$, respectively) are odd. Otherwise it is usually equal to 2, with few exceptions (for $d_1 = -4$ and $d_2 = -51, -123$, or $d_2 = -132$ we couldn't find a better regulator lattice for $F_4 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ to make the index quotient 2).

We give an explicit example for $F = \mathbb{Q}(\sqrt{-11}, \sqrt{-19})$. With subfields $F_1 = \mathbb{Q}(\sqrt{-11})$ and $F_2 = \mathbb{Q}(\sqrt{-19})$, we get

$$\widetilde{R}_2(F) \doteq 528.23, \quad \widetilde{R}_2(F_1) \doteq 16.59, \quad \text{and} \quad \widetilde{R}_2(F_2) \doteq 31.83,$$

with quotient 1. Here we know (from Tate [21] and Skałba [20]) that both groups $K_2\mathcal{O}_{F_j}$ $(j = 1,2)$ are trivial.

Similarly, if $F_1 = \mathbb{Q}(\sqrt{-571})$ (still considering $F_2 = \mathbb{Q}(\sqrt{-19})$), where we know from [2] that $K_2\mathcal{O}_{F_1} = \mathbb{Z}/5\mathbb{Z}$, then

$$\widetilde{R}_2(F) \doteq 32315.473, \quad \widetilde{R}_2(F_1) \doteq 1015.004, \quad \text{and} \quad \widetilde{R}_2(F_2) \doteq 31.83,$$

still leaving the quotient 1.

If we consider instead $F_1 = \mathbb{Q}(\sqrt{-23})$, for which $K_2\mathcal{O}_{F_1} = \mathbb{Z}/2\mathbb{Z}$ has *even* order, we find

$$\frac{\widetilde{R}_2(F)}{\widetilde{R}_2(F_1)} \doteq \frac{2463.935}{38.695} \doteq 2 \cdot \widetilde{R}_2(F_2),$$

whereas for $F_1 = \mathbb{Q}(\sqrt{-51})$, for which also $K_2\mathcal{O}_{F_1} = \mathbb{Z}/2\mathbb{Z}$, we obtain

$$\frac{\widetilde{R}_2(F)}{\widetilde{R}_2(F_1)} \doteq \frac{2409.997}{75.695} \doteq \widetilde{R}_2(F_2).$$

The former case is the more common one when considering that quotient in the case that at least one of the K_2-orders is even, but the latter case also occurs for certain distinguished discriminants.

Let us recall that for an imaginary quadratic field $F = \mathbb{Q}(\sqrt{-d})$, the order of $K_2(\mathcal{O}_F)$ is odd iff $d = 1, 2, p$ or $2p$, where $p \equiv \pm 3 \pmod 8$ is a prime, see [8].

12.4. Dihedral cases

The case $p = 3$. We looked at several number fields of the form $F = \mathbb{Q}(\sqrt[3]{d}, \sqrt{-3})$ which have Galois group $D_6 \cong S_3$. As a typical case, take $d = 2$: we find, using the notation $F_0 = \mathbb{Q}(\sqrt{-3})$ for the unique quadratic subfield and $F_1 = \mathbb{Q}(\sqrt[3]{d})$ for one of the three isomorphic cubic subfields of F :

$$\begin{aligned} \widetilde{R}_2(F) &\doteq 389.3591874/\pi^3 &&\doteq 12.55743124, \\ \widetilde{R}_2(F_1) &\doteq 13.84967835/\pi &&\doteq 4.408489539, \\ \widetilde{R}_2(F_0) &\doteq 2.029883212/\pi &&\doteq 0.646131894, \end{aligned}$$

from which we find for the quotient (see Theorem 2)

$$\frac{3}{Q_2(F)} = \frac{\widetilde{R}_2(F)}{\widetilde{R}_2(F_1)^2\widetilde{R}_2(F_0)} \doteq 1.$$

The corresponding quotient for the fields $F = \mathbb{Q}(\sqrt[3]{d}, \sqrt{-3})$ with squarefree $d < 50$ is either 3 (for $d = 17, 19, 22, 23, 33, 34, 37$) or 1 (for the remaining d). The elements found by the program which conjecturally generate the corresponding Bloch group are typically too complicated to write down here.

The case $p \geq 5$. For $5 \leq p \leq 14,\ p \neq 12, 13$, we considered one polynomial each defining a Galois extension F_{2p} of $\mathbb{Q}$ with Galois group D_{2p}, as linked from the GP-PARI website.

- 1. *The case $p = 5$.* We consider the field F_{10} defined by the polynomial $\sum_{i=0}^{10} c_i x^i$, where $(c_i)_i = (1, 1, 2, -1, 10, -18, 20, -18, 12, -5, 1)$, of discriminant -47^5 and signature $(r_1, r_2) = (0, 5)$, which is Galois over $\mathbb{Q}$ with Galois group D_{10}.

  Its (up to an isomorphism) unique degree 5 subfield F_5 can be described by $\sum_{i=0}^{5} c_i x^i$, where $(c_i)_i = (-11, 5, -2, 7, -5, 1)$, and is of discriminant 47^2 and signature $(1, 2)$.

  Its unique degree 2 subfield F_2 is the imaginary quadratic field of discriminant -47.

  Magma gives $\zeta^*_{F_{10}}(-1) \doteq -3.75562$ and our program, written in GP-PARI, finds a conjectural dilogarithm regulator

  $$\widetilde{R}_2(F_{10}) \doteq 13791.5413/\pi^5 \doteq 45.06749724.$$

  Hence, since $w_2(F_{10}) = 24$, we are led to

  $$2\widetilde{R}_2(F_{10}) \doteq -w_2(F_{10})\zeta^*_{F_{10}}(-1).$$

  For explicit (and short) elements in the Bloch group of F_{10}, see [9], §5.1.1.

Similarly, we get $\zeta^*_{F_5}(-1) \doteq -0.091823$ and $\zeta^*_{F_2}(-1) \doteq -3.09308$, while the corresponding (conjectural) regulators found are

$$\widetilde{R}_2(F_5) \doteq 10.8753/\pi^2 \doteq 1.101898268,$$
$$\widetilde{R}_2(F_2) \doteq 116.606541/\pi \doteq 37.11701479,$$

so that

$$2\widetilde{R}_2(F_k) \doteq -w_2(F_k)\zeta^*_{F_k}(-1), \quad \text{for} \quad k = 2 \text{ or } 5.$$

(Again, $w_2(F_k) = 24$ in both cases.)

As a consequence, we obtain (see Theorem 2)

$$\frac{5}{Q_2(F_{10})} = \frac{\widetilde{R}_2(F_{10})}{\widetilde{R}_2(F_5)^2\widetilde{R}_2(F_2)} \doteq 1,$$

from which we should expect that $k_2(F) = k_2(F_5) = k_2(F_2) = 2$ (the latter identity of which has been shown, cf. [2]).

A slightly more interesting case is the D_{10}-field F'_{10} of discriminant $-2^{15}11^8$ given by $x^{10} + 6x^8 + 21x^6 + 12x^4 - 28x^2 + 32$ and its subfields F'_5 and F'_2. In this case we find

$$80\widetilde{R}_2(F'_{10}) \doteq -w_2(F'_{10})\zeta^*_{F'_{10}}(-1)$$

and

$$40\widetilde{R}_2(F'_5) \doteq -w_2(F'_5)\zeta^*_{F'_5}(-1),$$

as well as

$$\widetilde{R}_2(F'_2) \doteq -w_2(F'_2)\zeta^*_{F'_2}(-1).$$

This would suggest that $k_2(F'_{10}) = 80$, $k_2(F'_5) = 40$ and $k_2(F'_2) = 1$ (again, the latter identity is known; it has been proved long ago by Tate [21]). Furthermore, this constitutes the first candidate of a D_{10}-field whose second regulator quotient is different from 1:

$$\frac{5}{Q_2(F'_{10})} = \frac{\widetilde{R}_2(F'_{10})}{\widetilde{R}_2(F'_2)\widetilde{R}_2(F'_5)^2} = 5.$$

This example already arose a few years ago in discussion with A. Bartel who, in collaboration with de Smit, investigated related questions from a more elaborate point of view [1].

- 2. *The case* $p = 7$: We consider the field F_{14} of discriminant -223^7, given by the coefficients

  $$(9,-87,353,-819,1301,-1618,1648,-1379,971,-566,276,-107,33,-7,1),$$

  of its minimal polynomial and its subfields F_7 and F_2. Their respective zeta values at -1 are given by $\zeta^*_{F_{14}}(-1) \doteq -86878788.53919$, $\zeta^*_{F_7}(-1) \doteq 152.73946$ and $\zeta^*_{F_2}(-1) \doteq -25.861205$. We find for their tentative regulators the approximate values

  $$\begin{aligned}\widetilde{R}_2(F_{14}) &\doteq 18631911242.299834/\pi^7 \doteq 6168908.53919,\\ \widetilde{R}_2(F_7) &\doteq 4371.583358/\pi^3 \doteq 140.9902711 \quad \text{and}\\ \widetilde{R}_2(F_2) &\doteq 974.944466/\pi \doteq 310.334462,\end{aligned}$$

  which results in the formulas

  $$\begin{aligned}2\cdot 13^2\widetilde{R}_2(F_{14}) &\doteq -w_2(F_{14})\zeta^*_{F_{14}}(-1),\\ 2\cdot 13\widetilde{R}_2(F_7) &\doteq -w_2(F_7)\zeta^*_{F_7}(-1),\\ 2\widetilde{R}_2(F_2) &\doteq -w_2(F_2)\zeta^*_{F_2}(-1).\end{aligned}$$

  From this we obtain (cf. Theorem 2)

  $$\frac{7}{Q_2(F_{14})} = \frac{\widetilde{R}_2(F_{14})}{\widetilde{R}_2(F_7)^2\widetilde{R}_2(F_2)} \doteq 1.$$

  Moreover, we expect that $k_2(F_{14}) = 2\cdot 13^2$ and $k_2(F_7) = 2\cdot 13$.

- 3. *The case* $p = 8$: In this case, we get $w_2(F_r) = 48$ for $r \in \{4,8,16\}$ and $w_2(F_r) = 24$ for $r = 2$. Still the formulas agree with the ones for $p = 5$, i.e. putting

  $$q(F_r) := w_2(F_r)\zeta^*_{F_r}(-1)/\widetilde{R}_2(F_r), \tag{12.2}$$

  (which conjecturally agrees with $k_2(F_r)$), we find $q(F_r) = 2$ for all four fields F_r, $r|16$ $(r > 1)$ in question.

- 4. *The case* $p = 9$: Consider the field F_{18} of degree 18 and discriminant $-2^{12}\cdot 107^9$. In this case we get $\widetilde{R}_2(F_{18}) \doteq 1507145405664649.50892/\pi^9 \doteq 50559910878.40792$ and $\zeta^*_{F_{18}}(-1) \doteq -227519598952.835$, so that

  $$-w_2(F_{18})\zeta^*_{F_{18}}(-1)/\widetilde{R}_2(F_{18}) \doteq 2^2\cdot 3^3,$$

  and the corresponding quotients $q(F_r)$ for the subfields of F_{18} of degree 9, 6, 3 and 2 are conjecturally given by $2^2\cdot 3$, $2^2\cdot 3$, 2^2 and 3, respectively.

- 5. *The case* $p = 10$: For the field F_{20} of discriminant $2^{20} \cdot 47^{10}$, we find a regulator

$$\widetilde{R}_2(F_{20}) \doteq 601489603356159.6482/\pi^{10} \doteq 6422873936.691262$$

and a zeta value $\zeta^*_{F_{20}}(-1) \doteq 1.49867 \cdot 10^{10}$, resulting in

$$w_2(F_{20})\zeta^*_{F_{20}}(-1)/\widetilde{R}_2(F_{20}) \doteq 2^3 \cdot 7.$$

The same factor $2^3 \cdot 7$ occurs for the corresponding quotient for its subfield F_4 of degree 4, but apparently not for the other degrees, for which the quotient becomes equal to 2.

Hence we expect $k_2(F_{20}) = k_2(F_4) = 2^3 \cdot 7$, so the non-trivial part of $K_2(\mathcal{O}_{F_{20}})$ should be induced from $K_2(\mathcal{O}_{F_4})$.

- 6. *The case* $p = 11$: This case is for the field F_{22} of discriminant 167^{11} and its subfields F_{11} and F_2. It is very similar to the one for $p = 5$ above and yields the analogous formula

$$2\widetilde{R}_2(F_r) \doteq \pm w_2(F_r)\zeta^*_{F_r}(-1), \quad r = 2{,}11 \text{ or } 22$$

with respective regulator values

$$\doteq 778052998185977 72.5399, \quad 9899632.6249 \quad \text{and} \quad 793.9095.$$

Hence the regulator quotient equals 1.

- 7. *The case* $p = 14$: The corresponding quotients for Galois extension F_{28} with Galois group D_{28} and discriminant $2^{28} \cdot 101^{14}$ and for its subfields F_k of degree k are as follows: for F_{28}, F_{14}, F_7, F_4 and F_2, we find $q(F_k) \doteq 2^{12} \cdot 19$, 2^6,2^4, 19 and 1, respectively.

12.5. The alternating group A_4

We considered several complex fields with Galois group A_4, the alternating group on 4 letters. It turns out that in each case there is a degree 4 and a degree 6 subfield F_4 and F_6 (both with $r_2 = 2$) which have exactly the same regulator. The fact that their regulators differ by a rational factor is a consequence of the Brauer-Kuroda relations and Conjecture 1, see (11.4).

More precisely, one seems to have

$$\zeta_{F_3}(-1) \cdot \widetilde{R}_2(F_6) = -2\zeta^*_{F_6}(-1),$$

where F_3 is the totally real field of degree 3 which is a subfield of F_6 and

$$\zeta_{\mathbb{Q}}(-1) \cdot \widetilde{R}_2(F_4) = -2\zeta^*_{F_4}(-1).$$

Nevertheless, although the covolumes agree, the actual lattices do not – instead, e.g. the elements of $\Lambda_2(F_6)$ arise from taking $\mathbb{Q}$-linear combinations (with very small denominators) of the *rows* of $\Lambda_2(F_4)$, where different rows correspond to different embeddings (of the Bloch elements whose dilogarithm values generate the lattice).

For example, F_{12} given by $x^{12} + 6x^{10} - 11x^8 + 42x^6 - 30x^4 + 40x^2 + 1$ has the subfields F_6 defined by $x^6 - 4x^5 + 4x^4 - 4x^3 + 20x^2 - 16x - 8$ and F_4 defined by $x^4 - 4x^3 + 14x^2 - 28x + 21$. They have both dilogarithmic regulator $\widetilde{R}_2(F_r) \doteq 1127.145385/\pi^2 \doteq 114.203704545$ $(r = 4,6)$, while their special values at -1 are given by $\zeta^*_{F_6}(-1) \doteq 2.7191358$ and $\zeta_{F_4}(-1) \doteq 4.7584876$, respectively. We thus get the quotients $\widetilde{R}_2(F_6)/\zeta^*_{F_6}(-1) \doteq 42$ and $\widetilde{R}_2(F_4)/\zeta^*_{F_4}(-1) \doteq 24$, which matches the above displayed formulas because $\zeta_{F_3}(-1) = -1/21$.

The lattice $\Lambda_2(F_4)$ is generated by the two (column) vectors (27.115...,14.087...) and (−1.9487...,40.556...), each vector being indexed by the two complex places of F_4, while $\Lambda_2(F_6)$, also equipped with two complex places, is generated by (21.2526...,19.3039...) and (34.2806...,−21.8981...).

The lattices are (numerically) related as follows:

$$\Lambda_2(F_6)\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\Lambda_2(F_4).$$

The entries for both lattices arise from entries of (a specific column

$$c_{12} = (9.9747...,-3.1379...,-16.1659...,0.9743...,20.278...,31.227...)$$

of) the rank 6 lattice for F_{12} as linear combinations with coefficients of modulus ≤ 2 (e.g. the first entry 27.115... for F_4 equals $c_{12}[1] + c_{12}[2] + c_{12}[5]$).

This suggests a kind of symmetrization of the Bloch groups via descent on number fields (note that this should be a more general phenomenon than the Galois descent which is known for the associated K-groups, as F_6 is not a Galois extension).

We find similar such lattice correspondences for other A_4-extensions.

12.6. The symmetric group S_4

Moreover, we obtain a further relationship for certain S_4-extensions: for example, for the Galois closure (of degree 24 and Galois group $S_4(6d) = [2^2]S_3$ in GAP notation [13]) of the field with minimal polynomial $x^6 - x^4 - x^3 - x^2 + 1$, we find two subfields of signature (0,4) and (4,4), respectively, whose regulators $\doteq 788598.76/\pi^4$ agree while their lattices are different but can be transformed into each other using integer matrices, in a more complicated manner than for the previous example.

An observation: In all the examples above the regulator quotients $\frac{m}{Q_2(F)}$, when computed, are integers, where $m = 2, p$ or 4 for the Galois groups $\mathbb{Z}/2 \times \mathbb{Z}/2, D_{2p}$ and A_4, respectively. Thus $Q_2(F)$ takes only values dividing m, and in these examples it holds that $m|\#G$. Is this true in general?

Acknowledgments: We thank the anonymous referee for valuable comments. We are grateful to Professor Anthony Bak for his remarks, which helped us to improve the exposition of the paper.

REFERENCES

1. A. Bartel, B. de Smit, Index formulae for integral Galois modules, *J. of London Math. Soc.,* to appear
2. K. Belabas, H. Gangl, Generators and relations for $K_2\mathcal{O}_F$, *K-Theory* **31**(3) (2004), 195–230
3. S. Bloch, Higher Regulators, Algebraic K-Theory, and Zeta Functions of Elliptic Curves, *CRM Monograph Series* **11**, American Mathematical Society, Providence, Rhode Island, USA, 2000
4. W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comp.* **24** (1997), 235–265
5. R. Brauer, Beziehungen zwischen Klassenzahlen von Teilkörpern eines galoisschen Körpers, *Math. Nachr.* **4** (1951), 158–174
6. J. Browkin, Conjectures on the dilogarithm, *K-Theory* **3** (1989), 29–56
7. J. Browkin, J. Brzeziński, Kejian Xu, On exceptions in the Brauer-Kuroda relations, *Bull. Polish Acad. Sci. Math.* **59**(3) (2011), 207–214
8. J. Browkin, A. Schinzel, On Sylow 2-subgroups of $K_2\mathcal{O}_F$ for quadratic number fields F, *Journ. Reine Angew. Math.* **331** (1982), 104–113
9. D. Burns, R. de Jeu, H. Gangl, On special values in higher algebraic K-theory and the Lichtenbaum-Gross conjecture, *Advances in Math.* **230**(3) (2012), 1502–1529
10. J.-L. Cathelineau, Remarques sur les différentielles des polylogarithmes uniformes, *Ann. Inst. Fourier, Grenoble* **46**(5) (1996), 1327–1347
11. H. Cohen, A Course in Computational Algebraic Number Theory (3rd corrected printing), *Graduate Texts in Mathematics* **138**, Springer Verlag, Berlin, 1996

12. A. Fröhlich, Artin root numbers and normal integral bases for quaternion fields, *Inventiones math.* **17** (1972), 143–166
13. The GAP group, GAP - Groups, Algorithms, and Programming, Version 4.4.12; 2008 (http://www.gap-system.org)
14. H. Hasse, Über die Klassenzahl Abelscher Zahlkörper, Akademie-Verlag, Berlin, 1952
15. S. Kuroda, Über die Klassenzahlen algebraischer Zahlkörper, *Nagoya Math. J.* **1** (1950), 1–10
16. J. Milnor, Introduction to algebraic K-theory, Princeton University Press, Princeton, N.J., 1971
17. W. Narkiewicz, Elementary and Analytic Theory of Algebraic Numbers, 3rd ed. Springer, Berlin, 2004
18. The PARI-group, PARI/GP, versions 2.1–2.4, Bordeaux, url http://pari.math.u-bordeaux.fr
19. P. Ribenboim, Classical Theory of Algebraic Numbers, Springer, New York, 2001
20. M. Skałba, Generalization of Thue's theorem and computation of the group $K_2\mathcal{O}_F$, *J. Number Theory* **46** (1994), 303–322
21. J. Tate, *Appendix to* The Milnor ring of a global field, in: *Algebraic K-theory, II: 'Classical' algebraic K-theory and connections with arithmetic* (Proc. Conf. Seattle, Wash., Battelle Memorial Inst., 1972), *Lecture Notes in Math.* **342** (1973), 349–446
22. A. Wiles, *The Iwasawa conjecture for totally real fields, Ann. of Math.* **131** (1990), 493–540
23. Zhou Haiyan, The tame kernel of multiquadratic number fields, *Comm. Algebra* **37**(2) (2009), 630–638
24. Zhou Haiyan, Higher class numbers in extensions of number fields, (submitted) (2012)

JERZY BROWKIN `browkin@impan.pl`

Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8, PL-00-956 Warsaw
Poland

HERBERT GANGL `herbert.gangl@durham.ac.uk`

Department of Mathematical Sciences
South Road, University of Durham
Durham, DH1 3LE
United Kingdom

Received: September 17, 2011