ALGEBRA II LECTURE NOTES
EPIPHANY TERM 2014

1. Quick motivation and overview

Motivation. The notion of a **group** is absolutely central and ubiquitous to mathematics, be it for linear algebra (e.g. matrix groups), geometry (e.g. symmetry/isometry groups of regular solids or polygons; Möbius transformations of the complex plane), mathematical physics (e.g. the Lorentz group of affine transformations in space-time), topology (e.g. the fundamental group of a torus, or more generally of any topological space), number theory (e.g. the set of integer solutions $(x, y) \in \mathbb{Z}^2$ of Pell's equation $x^2 - dy^2 = 1$, where $d \in \mathbb{Z}_{>0}$), Galois theory (e.g. Galois groups of field extensions) or algebraic geometry (e.g. rational solutions $(x, y) \in \mathbb{Q}^2$ of the elliptic curve $x^3 + y^3 = p$ for a prime $p \equiv 4 \pmod 9$)).

Overview. We give an outline of the topics that we will treat in this part of the course:

— Revision and introduction of structural properties and of important families of groups (e.g. S_n, A_n or D_n);
— Tools to distinguish groups from each other (numerical invariants, structural invariants);
— Methods to relate or even identify groups (homomorphisms, isomorphisms);
— How to break up a group into smaller pieces (distinguished subgroups, quotient groups);
— Conversely, how to splice groups together (direct product [maybe also semi-direct product]);
— Methods to "visualise" groups ("action" of a group on a set);
— Classification theorems (e.g. classification, for p a prime, of all groups of order p^2, classification of (finitely generated) *abelian* groups;
— Structural theorems ("Orbit-Stabiliser", "Sylow", "Cauchy" [if $p \mid \#G$ then $\exists$ subgroup of G of order p]).

2. Reminders from last term

In Michaelmas term, a number of properties have already been discussed, we summarise a few important ones here.

Recall that a *subgroup* H of a group G is a non-empty subset of G that is closed under composition and under taking inverses. We then denote this fact by $H < G$ (rather than just by $H \subset G$). (Examples are $n\mathbb{Z} < \mathbb{Z}$ for any $n \in \mathbb{Z}$, or $\mathbb{Q}^* < \mathbb{R}^* < \mathbb{C}^*$, where the R^* denotes the *units* of the ring R which constitute a group by themselves. There are always obvious subgroups (called "trivial"): $\{e\} < G$ and $G < G$.)

There are subgroups of a group G *generated by* an element of G, and denoted by diamond brackets: for a subset consisting of a single element g, one puts

$$\langle g\rangle = \{g^n \mid n \in \mathbb{Z}\}\,.$$

More generally, the subgroup generated by a subset $S \subset G$ consists of all the *finite* products of elements in S and of their inverses, a non-trivial example being, for $S = \{\frac{1}{2}, 3, 7\} \subset \mathbb{Q}^*$,

$$\langle S\rangle = \{2^m 3^n 7^r \mid m, n, r \in \mathbb{Z}\}\,.$$

Recall that the *order* of an element $g \in G$ is the smallest *positive* integer r such that $g^r = e$, the identity element in G, provided such an r exists; otherwise the order of g is ∞. The order of an element always divides the group order $\#G$.

Moreover, an important theorem of Lagrange states something more general: $H < G \Rightarrow \#H \mid \#G$.

Here's a somewhat more lyrical way to express this:

Lagrange's size insights

Take a subgroup, say H, of a given group G
with their sizes respectively called s and t.
Old Lagrange has refuted that "size does not matter",
as the former one clearly divideth the latter.

H.G.

Recall that a *normal* subgroup $H < G$ (denoted $H \lhd G$) is characterised by its satisfying $gHg^{-1} \subset H$ for any $g \in G$; equivalently, $gHg^{-1} = H$ for any $g \in G$; also equivalently, $gH = Hg$ for any $g \in G$ (i.e. each left coset of H is also a right coset of H); yet another equivalent way to phrase it is $ghg^{-1} \in H$ for any $h \in H$, $g \in G$. Normal subgroups are important, as they allow to give the set of cosets $gH = \{gh \mid h \in H\}$ the structure of a group: multiplying the cosets (with respect to H) of g and g' gives $(gH)(g'H) = (gg')H$, the coset of gg'; also, the inverse of gH is simply $g^{-1}H$. (Note that this multiplication does not make sense if H is *not* normal!)

As a consequence, we can write G/H as the quotient group for H normal (it consists precisely of the cosets in G w.r.t. H).

3. Conjugacy classes and the centre

An important notion closely connected with the one of a normal subgroup is the one of conjugacy. We will get a first glimpse in this section and will revisit the notion in due course.

Proposition. *Let H be a subgroup of G. Then we have*

$$H \text{ is normal in } G \quad\Leftrightarrow\quad H \text{ is a union of conjugacy classes of } G\,.$$

Proof. "$\Rightarrow$": if H is normal in G then, by definition of being normal, whenever $h \in H$ we also have $ghg^{-1} \in H$ for any $g \in G$. But this means that $\{ghg^{-1} \mid g \in G\}$, the conjugacy class of h in G, is a subset of H. So we can write $H = \bigcup_{h\in H} h \subset \bigcup_{h\in H}\{ghg^{-1} \mid g \in G\}$.

Now it remains to note that the latter expression is indeed a union of conjugacy classes, that it obviously contains H, but also that it is contained in H (any of the individual $\{ghg^{-1} \mid g \in G\}$ does), so it actually agrees with H.

"$\Leftarrow$": Suppose the subgroup H is the union of certain conjugacy classes in G. Then we have to show that $gHg^{-1} = H$ or, what is actually equivalent, $gHg^{-1} \subset H$. But

$$gHg^{-1} = \bigcup_{h \in H} ghg^{-1} \subset \bigcup_{h \in H} \{ghg^{-1} \mid g \in G\} = H\,.$$

In the last equality we have used that H is the union of conjugacy classes (necessarily the conjugacy classes of all its elements). □

Example: (Conjugacy classes of D_3)
There are three conjugacy classes in the dihedral group D_3, which can be viewed as the group of symmetries of an equilateral triangle in the plane. It consists of 6 elements: the identity e, two non-trivial rotations r and r^2 (around $2\pi/3$ and $4\pi/3$, respectively) and three rotations s, rs and r^2s (around the respective axes defined by the vertices of the triangle and their opposite medians).

Recall that we have the following three basic relations among r and s (which are complete in that they imply any relation among r and s):
$r^3 = e$, $s^2 = e$ and $srs^{-1} = r^2$.

(1) The conjugacy class of e is simply $\{e\}$, since $geg^{-1} = e$ for any $g \in D_3$.
(2) The conjugacy class of r is $\{r, r^2\}$: we write

$$\{grg^{-1} \mid g \in D_3\} = \{ere^{-1}, rrr^{-1}, r^2rr^{-2}, srs^{-1}, (sr)r(sr)^{-1}, (sr^2)r(sr^2)^{-1}\}$$

where the first three elements agree with r and the last three with r^2.
(3) The conjugacy class of s is $\{s, rs, r^2s\}$: we write

$$\{gsg^{-1} \mid g \in D_3\} = \{ese^{-1}, sss^{-1}, r^2sr^{-2}, (sr)s(sr)^{-1}, rsr^{-1}, (sr^2)s(sr^2)^{-1}\}$$

where the first two elements are equal to s, the following two equal to rs and the final two equal to r^2s.

Overall, we see that D_3 partitions into 3 conjugacy classes of size 1, 2 and 3, respectively.

Proposition. *Conjugate elements of a group G have the same order.*

Proof. Compare $x \in G$ and $gxg^{-1} \in G$ for an arbitrary $g \in G$. First note that

$$(gxg^{-1})^n = \underbrace{(gxg^{-1})(gxg^{-1}) \cdot \cdots \cdot (gxg^{-1})}_{n \text{ blocks}} = gx^ng^{-1}$$

as the intermediate $g^{-1}g$ drop out.
Now show that $(gxg^{-1})^n = e \;\Leftrightarrow\; x^n = e$, which then implies the claim (the "order" of an element is the smallest positive such n). Indeed,

$$e = (gxg^{-1})^n = gx^ng^{-1} \quad \Leftrightarrow \quad g = gx^n \quad \Leftrightarrow \quad e = x^n\,. \qquad \square$$

Example. For the case $G = D_3$, we have $\mathrm{ord}_{D_3}(r) = 3 = \mathrm{ord}_{D_3}(r^2)$ and $\mathrm{ord}_{D_3}(s) = 2 = \mathrm{ord}_{D_3}(rs) = \mathrm{ord}_{D_3}(r^2s)$.

Remark. Let G be a group. Then G is abelian if and only if all the conjugacy classes consist of a single element.

Proof. For G abelian and any $x \in G$ we have $\{gxg^{-1} \mid g \in G\} = \{gg^{-1}x \mid g \in G\} = \{x \mid g \in G\} = \{x\}$. Conversely, if a conjugacy class $\{gxg^{-1} \mid g \in G\}$ consists of a single element, that means that this element must be x (specialise $g = e$, for example) and hence we must have in particular $gxg^{-1} = x$, i.e. $gx = xg$, i.e. x commutes with any element in G. As x was arbitrary, this shows that any two elements of G commute, so G is indeed abelian. □

Another important notion is the *centre* of a group G, which consists of those elements in G which commute with all the other elements in G (they clearly commute with themselves, anyway). The centre turns out to be a group itself.

Definition. The **center** $Z(G)$ of a group G is defined by

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}\,.$$

Example. (1) The center of D_3 can neither contain r nor s, as $rs \neq sr$. For similar reasons, it cannot contain r^2, rs or r^2s. We conclude that $Z(D_3) = \{e\}$.

(2) The center of a cyclic group $\langle g\rangle$ is the group itself, as any g^i commutes with any g^j. (This uses that the addition for the exponents (in $\mathbb{Z}$) is commutative.)

Proposition. *The center $Z(G)$ of a group G is a normal subgroup of G.*

Proof. We first verify that $Z(G)$ is indeed a subgroup (which is not quite obvious from the way it is defined).
Let x and y be in $Z(G)$, i.e. $xg = gx$ and $yg = gy$ for any $g \in G$.
Then $xy \in Z(G)$ as well: $(xy)g = xgy = g(xy)$.
Also x^{-1} is in the centre: from inverting both sides of $xg = gx$ for all $g \in G$ we find $g^{-1}x^{-1} = x^{-1}g^{-1}$ for all g, but with g also g^{-1} runs through G.
Moreover, for each $x \in Z(G)$ we have that its conjugacy class $x^G = \{gxg^{-1} \mid g \in G\}$ equals $\{x\}$ (cf. above remark). In particular $Z(G)$, obviously equal to the union of its elements, is also equal to the union of the corresponding conjugacy classes. By one of the above propositions we find that $Z(G)$ is normal in G. □

Examples.

(1) The centre of an *abelian* group is the group itself. ⟦Clearly, every element is in the centre as it commutes with any other element.⟧

(2) A more ambitious example is the group $G = \mathrm{GL}_2(\mathbb{R})$. The condition to commute with all the other matrices in G can be pinned down by looking at specific matrices, e.g. $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and its transpose. Equating

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

implies that we must have $c = 0$ and $a = d$.
In a similar way, we find that $b = 0$ must hold (use the inverse of g above).

Conversely, we can easily see that any matrix satisfying these three conditions $c = 0$, $a = d$ and $b = 0$, i.e. which is of the form $a \cdot \mathrm{Id}$ for Id the 2×2–identity matrix, does indeed commute with every other matrix (all entries are simply multiplied by a when multiplying with $a \cdot \mathrm{Id}$ either on the left or on the right).

Conclusion: $Z(\mathrm{GL}_2(\mathbb{R})) = \{a \cdot \mathrm{Id} \mid a \neq 0\}$. (Note that the zero matrix does not lie in $\mathrm{GL}_2(\mathbb{R})$.)

Aside. The last example gives rise to an interesting quotient: since $Z(G)$ is a normal subgroup of G, we can always form the quotient group $G/Z(G)$. In the case of an abelian group, this quotient is the trivial group, while in the case of D_3 the quotient is isomorphic to D_3 itself.
For $G = \mathrm{GL}_2(\mathbb{R})$, the quotient can be identified with the so-called *fractional linear transformations* of the complex numbers: a typical fractional linear transformation looks as follows: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, the map $x \mapsto \dfrac{ax+b}{cx+d}$ defines a transformation of the complex numbers (minus the real numbers, to make sure it is well-defined: we want to avoid $x = -d/c$ which would introduce a pole) into themselves. This assignment provides a group homomorphism (of a matrix group to a group of functions) with kernel precisely the center $Z(\mathrm{GL}_2(\mathbb{R}))$ (the diagonal entries cancel in the fraction).

4. Permutation groups

How can we actually "pin down" a group? One of the most important sets of groups is formed by permutation groups. In fact, we will see that, in a sense, any group can be viewed as some kind of permutation group. This will often enable us to get a reasonable grip on a group (or rather on its objects).

Definition. A **permutation** of a non-empty set X is a bijection (i.e. injective and surjective map) from X to itself.

Notation. For X a non-empty set, we put

$$S_X = \{\text{bijections} : X \to X\}.$$

Fact. $(S_X, \circ)$ becomes a group where the binary operation "$\circ$" is the composition of functions.
⟦Associativity holds for composition of functions in general, the identity element of that group is simply the identity function on X, and the inverse of a bijection is given by reversing the association of objects: if $\sigma(g_i) = g_i'$, then for σ^{-1} we have $\sigma^{-1}(g_i') = g_i$.⟧

In particular, we put $S_n := S_{\{1,\ldots,n\}}$ for $n \geq 1$, the usual symmetric group on n letters.

Lemma. $\#S_n = n!$ for any $n \geq 1$.
⟦How many choices do we have for a bijection $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$? Fix the image of "1" (we have n choices), then the image of "2" (only $n-1$ choices left), ..., then finally the image of n (only one choice). ⟧

Notations and definitions. Any permutation of $\{1,\ldots,n\}$ can be more concisely written by inserting the image of each element below it: for instance the permutation $\sigma : \{1,2,3\} \to \{1,2,3\}$ given by $\sigma(1) = 3,\ \sigma(2) = 1,\ \sigma(3) = 2,$ will often be written as

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Specific permutations in S_n are **cycles of length** k or k**-cycles** ($1 \leq k \leq n$), which are bijections for a given *subset* $\{i_1,\ldots,i_k\}$ of size k of $\{1,\ldots,n\}$ as follows:

$\sigma(i_1) = i_2, \quad \sigma(i_2) = i_3, \quad \ldots \quad \sigma(i_{k-1}) = i_k, \quad \sigma(i_k) = i_1.$
We will write such a k-cycle in the above notation as

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_2 & i_3 & & i_1 \end{pmatrix},$$

or even more concisely as

$$(i_1\, i_2\, \ldots\, i_k).$$

Note that this is not unique, we could have also written it as $(i_2\, i_3\, \ldots\, i_k\, i_1)$ or $(i_3\, i_4\, \ldots\, i_1\, i_2)$ etc., overall there are precisely k ways to write the cycle in that more concise form.

Cycles of length 2, i.e. of the form $(i_1\, i_2)$, are called **transpositions**.

Two cycles are called **disjoint** if their members do not intersect.

For example, the cycles $(1\,3\,5)$ and $(2\,4)$ in S_5 are disjoint, while $(1\,3\,5)$ and $(1\,2\,4)$ are not (they share the common member "1").

Facts.

(1) Disjoint cycles commute with each other.
(2) Every permutation is a product of *disjoint* cycles, and in an essentially unique way. ("Essentially" meaning: up to ordering the individual cycles and up to the k different ways to write a given k-cycle.)

⟦As to (1), bijections of two disjoint subsets of a given set do not affect each other; this applies in particular to the product of two disjoint cycles. As to (2), each bijection σ of $\{1, \ldots, n\}$ is subdivided into bijections of subsets; maybe think of a graph with n vertices labelled by 1, ..., n with two vertices i and j connected by a directed edge from i to j whenever $\sigma(i) = j$, then the disjoint cycles of σ correspond to the different components of the graph (there might be individual vertices as components). ⟧

Example. Write the following permutation as a product of disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 3 & 2 & 1 & 4 & 8 & 9 & 7 & 6 & 10 \end{pmatrix} = (1\,5\,4)(2\,3)(7\,9\,6\,8)(10).$$

Another way to write it in the cycle notation would be $(3\,2)(6\,8\,7\,9)(10)(5\,4\,1)$.

How to multiply two cycles? It is not completely obvious how to multiply two cycles. We compose the two corresponding bijections to a new bijection. (The notation we are using is slightly counterintuitive, as one needs to work "from right to left". Some authors use the opposite notation (going from left to right), but then they need to write functions on the right, i.e. $(x)f$ rather than $f(x)$, as we are used to.)

We give an example using the following permutations (of $\{1, \ldots, 5\}$) denoted σ and τ:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}.$$

Composing the two permutations $\sigma \circ \tau$ corresponds to applying τ first and then σ, i.e.

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}.$$

We can achieve this by first writing τ and then writing underneath σ, but rearranged in such a way as to let the top line of σ agree with the bottom line of τ:

$$\begin{matrix}\left(\begin{smallmatrix}1&2&3&4&5\\2&3&4&1&5\end{smallmatrix}\right)\\\left(\begin{smallmatrix}1&2&3&4&5\\4&3&1&5&2\end{smallmatrix}\right)\end{matrix}\quad\text{rearranged to}\quad\begin{matrix}\left(\begin{smallmatrix}1&2&3&4&5\\{\color{red}2}&{\color{red}3}&{\color{red}4}&{\color{red}1}&{\color{red}5}\end{smallmatrix}\right)\\\left(\begin{smallmatrix}{\color{red}2}&{\color{red}3}&{\color{red}4}&{\color{red}1}&{\color{red}5}\\3&1&5&4&2\end{smallmatrix}\right)\end{matrix}$$

and then simply drop the intermediate (red) rows altogether.

One important simplifying convention is to drop all the 1-cycles (j). So the cycle $(1\,3\,5)(2)(4)$ in S_5 will be henceforth denoted $(1\,3\,5)$ only—in general, if it is clear in which group S_n we are working then the missing 1-cycles can easily be reconstructed: simply add a 1-cycle for each number $\leq n$ missing in the product of cycles.
Moreover, we will drop the $\circ$ signs.

Examples. Multiply $\sigma = (1\,2)$ and $\tau = (1\,3)$ in S_3 to

$$\sigma \circ \tau = (1\,2)(1\,3) = \begin{matrix}\left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)\\\left(\begin{smallmatrix}3&2&1\\3&1&2\end{smallmatrix}\right)\end{matrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$$

and the latter can also be written in our even shorter notation as $(1\,3\,2)$.

Proposition.

(1) *Any $\sigma \in S_n$ can be written (also called "factored") as a product of transpositions.*
(2) *The parity of the number of transpositions needed in any factorization of $\sigma \in S_n$ is the same. In particular, this number is well-defined modulo 2.*
(3) *An element with disjoint cycles of lengths $k_1, \ldots k_m$ has order* $\operatorname{lcm}(k_1, \ldots, k_m)$.

Proof. (1) It suffices to write any given k-cycle ($k \geq 2$) as a product of transpositions. A possibility for the latter is as follows (cf. Sheet 2, Q1):

$$(1\,2\ \cdots\ k) = (1\,k)(1\ k-1)\ \cdots\ (1\,2)\,.$$

(2) For the second claim, one can introduce independent variables $x_1, \ldots, x_n$, and look at the expression

$$P_n = \prod_{1\leq i<j\leq n} (x_i - x_j)\,.$$

For any permutation $\sigma \in S_n$ we then consider

$$P_{n,\sigma} = \prod_{1\leq i<j\leq n} (x_{\sigma(i)} - x_{\sigma(j)})\,.$$

This quantity has the same factors as P_n, at least up to sign, and for a transposition $\sigma = (i\,j)$ we get $P_{n,\sigma} = -P_n$. Moreover, this procedure is multiplicative, so for σ a product of r transpositions we have $P_{n,\sigma} = (-1)^r P_n$.
For (3) first check the case $m = 1$, then show that any element raised to $L = \operatorname{lcm}(k_1, \ldots, k_m)$ indeed becomes the identity (use that disjoint cycles commute) and then show that any proper divisor of L (i.e. different from L) does not suffice.
Here we have used the following notation: the lcm (=least common multiple) of a set of integers is the smallest positive integer which is a multiple of each element in that set, e.g. $\operatorname{lcm}(6, 8, 10) = 120$. For two numbers, one has $\operatorname{lcm}(m, n) = n{\cdot}m/\gcd(n, m)$.
□

Each S_n has a distinguished subgroup, denoted A_n ("A" for "alternating"), which has half the size of S_n. We can characterise it using the following numerical invariant.

Definition. The sign of a permutation $\sigma \in S_n$ is defined as

$$\operatorname{sgn}(\sigma) = (-1)^t ,$$

where t denotes the number of transpositions needed in a factorization of σ.

Remark. By the previous proposition, the number t is well-defined modulo 2, hence sgn is indeed well-defined. We can obtain it in a slightly more economical way as follows: let $\sigma \in S_n$ be a permutation whose (essentially unique) cycle decomposition is a product of cycles of length $k_1, \ldots, k_r$. Then the **sign** of the permutation σ is given by

$$\operatorname{sgn}(\sigma) = (-1)^{(k_1-1)+\cdots+(k_r-1)} ,$$

i.e. $\operatorname{sgn}(\sigma)$ is equal to 1 if $\sum_{i=1}^{r} k_i$ has the same parity as r, and otherwise it is equal to -1.

Examples.

1. A transposition has the parity -1.
2. Any k-cycle has the parity $k-1$: write $(i_1\, i_2\, \ldots\, i_k) = (i_1 i_k)(i_1 i_{k-1}) \cdots (i_1\, i_2)$.

Lemma. For $n \geq 2$, the function sgn provides a surjective homomorphism of groups

$$\operatorname{sgn} : S_n \to \{\pm 1\} .$$

Proof. Suppose we can write σ_i as a product of t_i transpositions ($i = 1, 2$). We need to check that $\operatorname{sgn}(\sigma_1\sigma_2) = \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2)$ for any $\sigma_1, \sigma_2 \in S_n$. But this is simply a consequence of the fact that we can write $\sigma_1\sigma_2$ in terms of $t_1 + t_2$ transpositions by composing the t_1 transpositions for σ_1 with the t_2 transpositions for σ_2.
Surjectivity is obvious as there is at least one transposition in S_n. □

Definition. A permutation σ in S_n is called *even* if $\operatorname{sgn}(\sigma) = 1$, otherwise it is called *odd.*
The kernel of $\operatorname{sgn} : S_n \to \{\pm 1\}$ is called the *alternating group* A_n, i.e.

$$A_n = \{\sigma \in S_n \mid \sigma \text{ is even}\} .$$

Proposition.

1. *The group A_n is normal in S_n.*
2. $\#A_n = \dfrac{n!}{2}$.
3. *The group A_n is generated by 3-cycles.*

Proof. *(1) Clear, as A_n is the kernel of a group homomorphism.*
(2) Clearly multiplying an even permutation by a transposition gives an odd permutation and vice versa. So a given fixed transposition produces a bijection between even and odd permutations in S_n (and there are no others). This implies the statement.
(3) Write $\sigma \in A_n$ as a product of an even *number of transpositions*

$$(i_1\, j_1)(i_2\, j_2) \ldots (i_{2r}\, j_{2r}) .$$

Then, starting from the left, combine two successive transpositions:
Case 1 *(non-disjoint) can write $(i\, j)(j\, k) = (j\, k\, i)$;*
Case 2 *(disjoint) can write $(i\, j)(k\, \ell) = (i\, j)(j\, k)(j\, k)(k\, \ell) = (j\, k\, i)(k\, \ell\, j)$.* □

Examples (of subgroups of A_4 and S_4):

(1) Consider the group generated by the element $(1\,2)(3\,4) \in A_4$:

$$\langle (1\,2)(3\,4) \rangle = \{(1\,2)(3\,4), e\}\,.$$

This group is isomorphic to the only group of order 2 up to isomorphism), the cyclic group of that order.

(2) Similarly, considering the 3-cycle $(1\,2\,3)$ we find

$$\langle (1\,2\,3) \rangle = \{e, (1\,2\,3), (1\,3\,2)\}\,,$$

isomorphic to the cyclic group of order 3.

(3) Consider the group generated by two elements

$$\langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle = \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}\,.$$

which is isomorphic to the Klein 4-group.

(4) Subgroups in S_4 which are not in A_4 are, e.g., $\langle (1\,2) \rangle$ (cyclic of order 2), $\langle (1\,2\,3\,4) \rangle$ (cyclic of order 4) or $\langle (1\,2), (1\,2\,3) \rangle$ which is isomorphic to S_3 (we find the isomorphism from S_3 to this subgroup of S_4 simply by adding the 1-cycle (4) to each of the six permutations).

(5) A further subgroup of S_4 but not of A_4 is given by

$$\langle r = (1\,2\,3\,4)\,,\ h = (1\,2)(3\,4) \rangle\,,$$

which realises the symmetry group of a square, i.e. D_4; we can check $r^4 = e = h^2$ as well as $hrh^{-1} = r^{-1}$ and then we can also verify that all $r^i h^j$ for $0 \le i \le 3$, $0 \le j \le 1$ are mutually different.

5. Distinguishing and identifying groups

Although we have encountered the definition of a direct product of groups and of an isomorphism of groups, it is quite instructive to see how these notions can be used to identify or to distinguish groups.

Let us list a few very useful ideas for distinguishing two groups, i.e. to show that they are not isomorphic to each other.

An isomorphism preserves in particular

— the order of a group;
— the set of orders of elements (with multiplicity);
— the property of being abelian/non-abelian.

The former two can be categorised as "numerical invariants" of the group, while the latter could be called a "structural invariant".

Examples.

(1) S_3 and $\mathbb{Z}_6$ are not isomorphic.
There is an element of order 6 in $\mathbb{Z}_6$, but not in S_3 (orders there are 1, 2 or 3).

(2) Recall that A_4 has order $\frac{1}{2}4! = 12$, as does D_6, and both are not abelian. Could they be isomorphic?
The set of orders of elements in A_4 is 1, 2 or 3 (we can find eight 3-cycles and three products of two disjoint transpositions), but in D_6 there is an element of order 6.
So $A_4 \not\cong D_6$.

Recall that the direct (or Cartesian) product $G \times H$ of two groups G and H is simply given by the pairs (g, h) with $g \in G$ and $h \in H$. But there is a structure of group on this product: simply work component-wise, i.e. $(g, h) \circ_{G \times H} (g', h') = (g \circ_G g', h \circ_H h')$ where the subscript of a $\circ$ indicates in which group we take the composition. The identity element in $G \times H$ is then the pair of respective identity elements (e_G, e_H). Recall also that the number of elements in the product is simply the product of the number of elements in the groups from which we started.

Example. Consider $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{1}), (\overline{0}, \overline{2}), (\overline{1}, \overline{0}), (\overline{1}, \overline{1}), (\overline{1}, \overline{2})\}$.
Note that $(\overline{a}, \overline{b})$ denotes $(a \pmod 2, b \pmod 3)$, i.e. the bars have a different meaning in the first and second component!

Claim: This direct product is isomorphic to a group we know better: $\mathbb{Z}_6$. How can we show this?
We could cook up an explicit isomorphism as follows, but we will give a better "machinery" in the Theorem-Criterion below. Clearly, the latter group is generated by the single element $\overline{1} = 1 \bmod 6$. So we try to find a single generator of $\mathbb{Z}_2 \times \mathbb{Z}_3$ as well: indeed, $(\overline{1}, \overline{1})$ does it. One easily checks that all $(\overline{a}, \overline{a})$ $(0 \le a \le 5)$ are different ⟦if $(\overline{a}, \overline{a}) = (\overline{b}, \overline{b})$ then comparing the first component gives that 2 divides $b - a$ while comparing the second component yields that 3 divides it, so overall 6 divides $b - a$; but both a and b are between 0 and 5, so must agree⟧, hence we have listed all $2 \cdot 3$ elements of $\mathbb{Z}_2 \times \mathbb{Z}_3$. In fact, we have even described the isomorphism:

$$\begin{aligned} \varphi : \mathbb{Z}_6 &\rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3 \\ a \bmod 6 &\mapsto (a \bmod 2, a \bmod 3) \end{aligned}$$

and it is clear that this map respects the group laws, i.e. is a homomorphism: we have for any a, $b \in \mathbb{Z}$

$$\varphi(a \bmod 6 + b \bmod 6) = \varphi((a+b) \bmod 6) = ((a+b) \bmod 2, (a+b) \bmod 3)\,,$$

while

$$\varphi(a \bmod 6) + \varphi(b \bmod 6) = (a \bmod 2, a \bmod 3) + (b \bmod 2, b \bmod 3)\,.$$

Both right hand sides give the same element, as we add component-wise.
Conclusion: we have found a surjective homomorphism of groups of the same size. This already implies that we in fact have found a group *iso*morphism: we can just define the inverse map by "going backwards": for $(a \bmod 2, b \bmod 3)$ we can find a integer $0 \le c \le 5$ such that $(c \bmod 2, c \bmod 3) = (a \bmod 2, b \bmod 3)$ (see above), and then we map this to $c \bmod 6$ in $\mathbb{Z}_6$. More generally, we have

Theorem. *For $m, n \ge 1$ we have*

$$\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n \iff \gcd(m, n) = 1\,.$$

Proof. The implication "$\Leftarrow$" is actually a consequence of the Chinese Remainder Theorem for rings: Look at the ideal $(n)_{\mathbb{Z}} = n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$ in the ring $\mathbb{Z}$ and similarly at $(m)_{\mathbb{Z}}$ as well as $(mn)_{\mathbb{Z}}$, and realise that $\mathbb{Z}_n$ is the same as the factor ring (also called quotient ring) $\mathbb{Z}/n\mathbb{Z}$.
Now forget about the ring multiplication, i.e. pass from the *ring* $\mathbb{Z}_n$ (more precisely the triple $(\mathbb{Z}_n, +, \cdot)$) to the *group* $\mathbb{Z}_n$ (more precisely the pair $(\mathbb{Z}_n, +)$).
For the other implication we can assume that $d = \gcd(m, n) > 1$ and put $m' = m/d$

and $n' = n/d$. Then $\gcd(m', n') = 1$ and one can show that the order of any element in $\mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_{m'd} \times \mathbb{Z}_{n'd}$ is at most $m'n'd$:

$$m'n'd(\overline{a}, \overline{b}) = \big(\underbrace{m'd}_{=m}(n'\overline{a}), \underbrace{n'd}_{=n}(m'\overline{b})\big)\,,$$

and both components are indeed $\overline{0}$ in the respective groups.
But the group order of $\mathbb{Z}_m \times \mathbb{Z}_n$ is $mn = m'n'd^2 > m'n'd$, and a cyclic generator of it would have to have this order, which cannot exist as we just checked. □

Notation. For two subsets E_1, E_2 of a group G we put

$$E_1 \circ E_2 := \{e_1 \circ e_2 \mid e_1 \in E_1,\ e_2 \in E_2\}\,.$$

This allows us to formulate a very useful criterion for checking if a group is the direct product of two of its subgroups. In fact, the implication "$\Leftarrow$" in the above theorem can be proved easily using it.

Theorem-Criterion. *Let H and K be subgroups of a group G such that the following three conditions hold:*

(1) $H \circ K = G$*;*
(2) $H \cap K = \{e\}$*;*
(3) $hk = kh \quad \forall h \in H, \forall k \in K$*.*

Then we have

$$G \cong H \times K\,.$$

Examples.

(1) The Klein 4-group V is given by the 4-element set $V = \{e, a_1, a_2, a_3\}$ with the relations $a_i^2 = e$ $(1 \le i \le 3)$ and $a_i a_j = a_k$ if $\{i, j, k\} = \{1, 2, 3\}$ (*).
We will show that it is the direct product of two subgroups of order 2. Put $H_i = \{e, a_i\}$ $(1 \le i \le 3)$. Clearly each H_i is a subgroup $[\![a_i^{-1} = a_i$, so it is closed under taking inverses$]\!]$. In fact, there is only one group of order 2 up to isomorphism, and each H_i is isomorphic to it.
Moreover, $H_i \cap H_j = \{e\}$ if $i \neq j$, and e.g. $H_1 \cdot H_2 = \{e, a_1, a_2, a_1 a_2\}$, but this equals V as $a_1 a_2 = a_3$.
By (*), elements in H_1 and H_2 commute with each other, so we can apply the criterion to obtain

$$V \cong H_1 \times H_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2\,.$$

(2) We want to show that $D_6 \cong Z_2 \otimes D_3$.
Recall that D_6 is generated by two elements r and s of orders 6 and 2, respectively, with the further relation $(rs)^2 = e$ or, equivalently, $srs = r^{-1}$. One shows that it consists of 12 elements, which we can choose as written in the form $r^i s^j$ $(0 \le i \le 5,\ 0 \le s \le 1)$.
Choose the following two subgroups:
$H = \langle r^3 \rangle$, a subgroup of order 2, and
$K = \langle r^2, s \rangle = \{e, r^2, r^4, s, r^2 s, r^4 s\}$, a subgroup of order 6 which is a group generated by $\tilde{r} = r^2$ and s with the relation (induced from D_6) $s\tilde{r}s = \tilde{r}^{-1}$ which we can obviously identify with D_3 .

Let us check the three conditions of the criterion:

(a) Multiply each member of K from the left by r^3, this will produce the six elements in D_6 which are not in K.
(b) $H \cap K = \{e\}$ is clear.
(c) To show: $r^3 \cdot (r^{2j}s^i) = (r^{2j}s^i) \cdot r^3$ for any $0 \le j \le 2$, $0 \le i \le 1$.
But $sr^3 = r^{-3}s = r^3 s$, so any power of s commutes with r^3, as clearly does every power of r.

Conclusion: In light of our Theorem-Criterion we find $D_6 \cong H \times K \cong \mathbb{Z}_2 \times D_3$

Our next aim is to "uniformise" groups in a certain sense, in order to treat them all from a common point of view, if needed. In fact, we will write every group as a subgroup of some permutation group S_X (the bijections of some (non-empty) set X). In order to motivate this, let us consider a more geometric occurrence of groups.

Theorem. *The group of rotational symmetries of the unit cube in $\mathbb{R}^4$ is isomorphic to S_4.*

Proof (idea): The following rotations of the cube exist. (We can view any rotation as represented by an orthogonal 3×3–matrix, more precisely by an element γ of $SO_3(\mathbb{R})$, and from Linear Algebra we obtain that one of the eigenvalues of γ is 1, hence there is line through the origin which is fixed point-wise by γ. This will give our rotation axis.)

(i) Rotation axis through two opposite face centers by an angle $\pi/2$, π or $3\pi/2$ (and 0, of course).
This gives us $\frac{6}{2}$ (face pairs) $\cdot$ 3 (non-trivial rotations) = 9 non-trivial rotations.
(ii) Rotation axis through two opposite vertices by an angle $2\pi/3$ or $4\pi/3$ (and 0).
This gives us $\frac{8}{2}$ (vertex pairs) $\cdot$ 2 (non-trivial rotations) = 8 non-trivial rotations.
(iii) Rotation axis through two opposite edges by an angle π (and 0).
This gives us $\frac{12}{2}$ (edge pairs) $\cdot$ 1 (non-trivial rotations) = 6 non-trivial rotations.

Overall, we find $9 + 8 + 6 = 23$ non-trivial rotations; adding the trivial one, we get 24 such rotations.

We can now "realize" this group as a permutation group, in several different ways. For example, we can try to keep track of what is happening to an indicative subset of the cube, all elements should be somehow of a similar nature, for example the set $\mathcal{V}$ of its vertices; or else the set $\mathcal{F}$ of its faces; or else the set $\mathcal{E}$ of its edges.

In the first case, we will recover the rotation group of the cube as a subset of $S_{\mathcal{V}} \cong S_8$, in the second case as a subset of $S_{\mathcal{F}} \cong S_6$, and in the third case as a subset of $S_{\mathcal{E}} \cong S_{12}$.

An even more economical way ensues if we take the set $\mathcal{D}$ of principal diagonals of the cube, as we can recover the cube rotations as a subset of $S_{\mathcal{D}} \cong S_4$, and for reasons of size—both sets are of order 24—we get that the two must agree.

The above are all instances of the following general fact.

Theorem (Cayley): *Each group $(G, \cdot)$ is isomorphic to a subgroup of some permutation group $(S_X, \circ)$.*
In fact, we can take X to be the underlying set *G.*

Proof. To each element $g \in G$ we assign a permutation L_g (the "left translation by g") defined by

$$\begin{aligned} L_g : \qquad & G \to G \\ & h \mapsto gh \end{aligned}$$

⟦Check the claim that L_g is indeed a bijection:
• injectivity: if $L_g(h) = L_g(h')$, then $gh = gh'$, and by left cancellation (of g) in G we find $h = h'$;
• surjectivity: for any $k \in G$ we find $g^{-1}k$ whose image under L_g is indeed $L_g(g^{-1}k) = k$.⟧

Now put

$$G' = \{L_g \in S_G \mid g \in G\}\,,$$

i.e. collect all left translations by elements in $g \in G$. This forms (so far only) a sub*set* G' of S_G.

Claim: G' is indeed a *group* (in fact, a subgroup of $(S_G, \circ)$).
• G' is non-empty: the identity permutation L_e represents the identity element in S_G (multiplying by e leaves each element in G invariant).
• closure under composition: for each L_g and L_h in G' also have $L_g \circ L_h \in G'$ (here the composition $\circ$ is taken in S_G, i.e. this is a composition of bijections).
Indeed, it coincides with L_{gh}:

$$L_g \circ L_h(k) = L_g(hk) = ghk = L_{gh}(k) \quad \forall k \in G\,.$$

• G' is closed under taking inverses as $L_g^{-1} = L_{g^{-1}}$:

$$L_{g^{-1}} \circ L_g(k) = g^{-1}gk = k = L_e(k) \quad \forall k \in G\,.$$

This settles the claim.

So far we have shown that the map

$$\begin{aligned} \psi : \qquad & G \to G' \\ & g \mapsto L_g \end{aligned}$$

is a *homo*morphism of groups.

Claim: ψ is in fact an isomorphism.
⟦Surjectivity holds by construction—note that ψ is a map with target G', not S_G. Injectivity is straightforward, using right cancellation in G: suppose $L_g = L_h$, i.e. $L_g(k) = L_h(k)$ for any $k \in G$; then in particular can take $k = e$ and find $g = L_g(e) = L_h(e) = h$.⟧
This completes proof of the theorem.

Example. Consider the Klein 4-group $G = V = \{e, a_1, a_2, a_3\}$, where the elements a_i are subject to the relations $a_i^2 = e$, as well as $a_i a_j = a_k$ if $\{i, j, k\} = \{1, 2, 3\}$.

We want to show that G is isomorphic to a subgroup of the bijections S_X where $X = \{x_1 = e,\, x_2 = a_1,\, x_3 = a_2, x_4 = a_3\}$.

The proof of Cayley's Theorem suggests to take the following: if $g = a_1$, then

$$L_g = L_{a_1} : \qquad \begin{aligned} e &\mapsto a_1 \cdot e = a_1 \\ a_1 &\mapsto a_1 \cdot a_1 = e \\ a_2 &\mapsto a_1 \cdot a_2 = a_3 \\ a_3 &\mapsto a_1 \cdot a_3 = a_2 \,. \end{aligned}$$

Hence L_{a_1} simply corresponds to the permutation $(x_1\, x_2)(x_3\, x_4)$.

In a similar way, L_{a_2} corresponds to $(x_1\, x_3)(x_2\, x_4)$ and L_{a_3} corresponds to $(x_1\, x_4)(x_2\, x_3)$.

Now $G' = \{L_e, L_{a_1}, L_{a_2}, L_{a_3}\}$ forms a group by the theorem and is indeed a subgroup of $S_X \cong S_4$.

In the example of the group of rotations of a cube, we had found natural homomorphisms of that group into S_X where X had the cardinality 4, 6, 8 or 12. All of the above are instances of the following notion.

$(K\ L\ E)$

A theorem of Cayley
announces rather gaily:
Any group can be seen—how astute!—
as consisting of things that permute.

H.G.

Definition. An **action of a group G on a (non-empty) set X** is a homomorphism

$$\varphi : G \to S_X \,.$$

In other words, for each $g \in G$ we choose a permutation $\varphi(g)$ of the set X such that

$$\varphi(g) \circ \varphi(h) = \varphi(gh) \qquad \forall g, h \in G \,.$$

Note. We neither assume φ to be injective nor surjective.
We will also say "the group G acts on X".

Example. We give two rather different examples of actions of $\mathbb{Z}$ on $\mathbb{R}$.

(1) Let $(\mathbb{Z}, +)$ act on $\mathbb{R}$ by translation (using the usual addition in $\mathbb{R}$):

$$\psi : \qquad \begin{aligned} &\mathbb{Z} \to S_{\mathbb{R}} \\ &n \mapsto L_n : \mathbb{R} \to \mathbb{R} \,, \quad \text{where } \ L_n(r) = n + r. \end{aligned}$$

We check that this is indeed a group action: for any m, $n \in \mathbb{Z}$ we have

$$L_m \circ L_n(r) = L_m(n + r) = m + (n + r) \,,$$

on the other hand we have

$$L_{m+n}(r) = (m + n) + r \,.$$

Hence indeed $L_m \circ L_n = L_{m+n}$ by associativity in $\mathbb{R}$.
$[\![$Note the different binary operations in $\mathbb{Z} (< \mathbb{R})$ and in $S_{\mathbb{R}}$.$]\!]$

(2) Let $(\mathbb{Z},+)$ act on $\mathbb{R}$ by multiplication of its "parity" (using the usual ring multiplication in $\mathbb{R}$):

$$\begin{aligned} \varphi : \quad & \mathbb{Z} \to S_{\mathbb{R}} \\ & n \mapsto M_n : \mathbb{R} \to \mathbb{R}\,, \quad \text{where } \ M_n(r) = (-1)^n r. \end{aligned}$$

We check that this is indeed a group action: for any m, $n \in \mathbb{Z}$ we have

$$M_m \circ M_n(r) = M_m((-1)^n r) = (-1)^m((-1)^n r)\,,$$

on the other hand we have

$$M_{m+n}(r) = (-1)^{(m+n)} r\,.$$

Hence indeed $M_m \circ M_n = M_{m+n}$ by the usual exponentiation rules.

(3) A more geometric example is the following: we define a group action of $(\mathbb{Z}_4,+)$ on $X = \{\text{vertices } v_1,\dots,v_8 \text{ of a cube}\}$ by fixing an axis through two opposite face centres and denote by r the rotation by an angle of $\frac{\pi}{2}$. Then $\varphi : \mathbb{Z}_4 \to S_X$ induces the following permutations (after suitable labeling of the vertices): $\overline{1}$ maps to the permutation induced by the rotation r, i.e.

$$\begin{aligned} \overline{1} &\mapsto (v_1\, v_2\, v_3\, v_4)(v_5\, v_6\, v_7\, v_8) \\ \overline{2} &\mapsto (v_1\, v_3)(v_2\, v_4)(v_5\, v_7)(v_6\, v_8) \\ \overline{3} &\mapsto (v_4\, v_3\, v_2\, v_1)(v_8\, v_7\, v_6\, v_5) \end{aligned}$$

while the identity in $\mathbb{Z}_4$, i.e. $\overline{0}$, of course maps to the identity permutation $e = (v_1)(v_2)(v_3)(v_4)(v_5)(v_6)(v_7)(v_8)$ in S_X.

In this last example we have seen that, for any of the images, the v_i for $i = 1,\dots,4$ never mingle with the ones for $i = 5,\dots,8$. So in a sense we have taken a set X of "unnecessarily large" size, as we could have easily made do with $v_1 \dots, v_4$ and would have obtained almost the same assignment as above except that we would simply forget $v_5,\dots,v_8$.

Definition. Let $\varphi : G \to S_X$ be a group action (of G on the set X), then for any $x \in X$ define

(1) $G(x) := \{ \underbrace{\varphi(g)}_{\text{a permut.}} (x) \mid g \in G\}$, called the ($G$–)**orbit** of x inside X;

(2) $G_x := \{g \in G \mid \varphi(g)(x) = x\}$, called the **stabiliser** of x in G.

Lemma. Any G_x is a subgroup of G.

Proof. • G_x is non-empty: $\varphi(e)$, the identity permutation, clearly fixes any $x \in X$; hence $e \in G_x$.

• G_x is closed under taking products: let $g, h \in G_x$, show $gh \in G_x$. $[\![\varphi(g)(x) = \varphi(h)(x) = x$ imply $\varphi(g)\big(\underbrace{\varphi(h)(x)}_{=x}\big) = \varphi(g)(x) = x$, whose left hand side is $\varphi(gh)(x)$ since φ is a homomorphism.$]\!]$

• G_x is closed under taking inverses: for $g \in G_x$ show $g^{-1} \in G_x$. $[\![\varphi(g^{-1})(x) = \varphi(g^{-1})\big(\underbrace{\varphi(g)(x)}_{=x}\big) = \varphi(g^{-1}g)(x) = x.]\!]$ $\square$

Example (revisited).

(1) Let $G = \mathbb{Z}$ act on $X = \mathbb{R}$ by translation as above.

$$\begin{aligned} \psi : \quad & \mathbb{Z} \to S_{\mathbb{R}} \\ & n \mapsto L_n : \mathbb{R} \to \mathbb{R}\,, \quad L_n(r) = n + r\,. \end{aligned}$$

Find the orbits and stabilisers under this action:
for any $x \in \mathbb{R}$ we get its *orbit* as

$$G(x) = \{\psi(n)(x) \mid n \in \mathbb{Z}\} = \{n + x \mid n \in \mathbb{Z}\} \subset \mathbb{R}\,;$$

and its *stabiliser* as

$$G_x = \{n \in \mathbb{Z} \mid n + x = x\} = \{0\}\,.$$

(2) $G = \mathbb{Z}$ acts on $X = \mathbb{R}$ via

$$\begin{aligned} \varphi : \quad & \mathbb{Z} \to S_{\mathbb{R}} \\ & n \mapsto \varphi(n) : \mathbb{R} \to \mathbb{R}\,, \quad \varphi(n)(r) = (-1)^n r \end{aligned}$$

and gives rise to *orbits*

$$G(x) = \{\varphi(n)(x) \mid n \in \mathbb{Z}\} = \{(-1)^n x \mid n \in \mathbb{Z}\} = \{x, -x\}\,.$$

Case $x \neq 0$: this set has two elements.
Case $x = 0$: this set has a single element.
Stabilisers: $G_x = \{n \in \mathbb{Z} \mid \varphi(n)(x) = x\} = \{n \in \mathbb{Z} \mid (-1)^n x = x\}$.
Case $x \neq 0$: $G_x = \{n \in \mathbb{Z} \mid n \text{ even}\} = 2\mathbb{Z}$.
Case $x = 0$: $G_0 = \{n \in \mathbb{Z}\} = \mathbb{Z}$.

(3) In our more geometric example let G be the rotations of the cube around a fixed axis through two opposite face centres (at left and right, say) and, for a change, X the *edges* of a cube, we find three orbits: for x any edge "on the left": $G(x)$ consists of all edges on the left, similarly for the edges "on the right", and for the edges "in the middle".
• All orbits are of size 4.
• The stabilisers are all $G_x = \{e\}$, as no edge is fixed by any of the non-trivial rotations.

(4) Check for yourself the following example: Let $\mathbb{R}$ act on $\mathbb{C}$ by letting $r \in \mathbb{R}$ act as the rotation $\varphi(r) : \mathbb{C} \to \mathbb{C}$ mapping x to $\varphi(r)(x) := e^{ir}x$. What are the orbits and stabilisers for a given $x \in \mathbb{C}$ (treat $x = 0$ separately)?
[Note that the orbits under this action probably agree with the colloquial meaning of "orbits" (e.g. of planets around a star etc.).]

The above is a rather clumsy notation, so we introduce an important *shortcut*: We usually leave out the homomorphism $\varphi : G \to S_X$ in the notation when we compute with group actions, so we will replace

$$\varphi(g)(x) \qquad \text{simply by} \qquad g(x) \quad \forall g \in G, \forall x \in X\,.$$

In particular, we rewrite

$$G_x = \{g \in G \mid g(x) = x\} \quad \text{and} \quad \varphi(g)\big(\varphi(h)(x)\big) = g\big(h(x)\big).$$

Proposition. *Let G act on a set X (and $\varphi : G \to S_X$ be the action). Then the distinct orbits $G(x)$ where x runs through X, partition X, i.e.*

(1) *each orbit is a non-empty subset of X;*
(2) *the union of all orbits is the whole set X;*
(3) *orbits are either disjoint or they coincide.*

Proof.

(1) Clearly $\varphi(e)$ is the identity permutation, so $G(x)$ must contain $\varphi(e)(x)$, i.e. x itself.

(2) Any $x \in X$ is in at least one orbit (in fact, in $G(x)$).

(3) Suppose $z \in G(x) \cap G(y)$ for some $x, y \in X$, in particular we can write $z = g_1(x)$ and $z = g_2(y)$. Then

$$x = g_1^{-1}\big(g_1(x)\big) = g_1^{-1}\big(g_2(y)\big) \in G(y)\,.$$

What is more, *any* $w \in G(x)$ also lies in $G(y)$:
$w \in G(x)$ means $w = g_3(x)$ for some $g_3 \in G$, so $w = g_3(x) = g_3\big(g_1^{-1}(g_2(y))\big) = (g_3 g_1^{-1} g_2)(y) \in G(y)$.
Hence $G(x) \subset G(y)$, and swapping roles of x and y we obtain the reverse inclusion.
Conclusion: $G(x) = G(y)$.

Remark. To be in the same orbit under a group action defines an equivalence relation.

There are two important ways in which a group G acts *on itself*, i.e. we can put $X = G$.

(1) by left translation (as in the proof of Cayley's Theorem):
$g \in G$ acts on $h \in G$ by $g(h) = gh$.
The orbit of any h is given by $G(h) = \{gh \mid g \in G\} = G$.
The stabiliser of any h is given by $G_h = \{g \in G \mid \underbrace{g(h)}_{=gh} = h\} = \{e\}$.

(2) by conjugation:
Here we have the homomorphism $\varphi : G \to S_G$ sending $g \in G$ to the bijection

$$\varphi(g) : \qquad \begin{aligned} &G \to G \\ &h \mapsto ghg^{-1}\,. \end{aligned}$$

Using our new shorthand, this expresses as follows: $g \in G$ acts on $h \in X (= G)$ by

$$g(h) = ghg^{-1}\,.$$

Check: this really gives a homomorphism.
$[\![gg'(h) = (gg')h(gg')^{-1} = g\big(g'hg'^{-1}\big)g^{-1} = g\big(g'(h)\big).]\!]$

Note that here the parentheses in red have a different meaning from the parentheses in black.

Conjugacy (and normality) revisited

Recall that two elements g and g' in a group G are *conjugate* (to each other) if there is an $h \in G$ such that $g' = hgh^{-1}$. The above example shows that a group acts on itself by conjugation. Hence we are led to

Definition. The orbit under conjugation of $g \in G$ is called the **conjugacy class** of g (in G), and is denoted by $\mathrm{ccl}_G(g)$:

$$\mathrm{ccl}_G(g) := \{hgh^{-1} \mid h \in G\}\,.$$

Examples.

(0) The set $\{e\}$ consisting of the identity element e in a group G forms a conjugacy class of its own:

$$\begin{aligned} G(e) &= \{g(e) \mid g \in G\} \\ &= \{geg^{-1} \mid g \in G\} \\ &= \{e \mid g \in G\} = \{e\}\,. \end{aligned}$$

(1) In an abelian group G, any conjugacy class is of size equal to 1: fix $g \in G$, then

$$\begin{aligned} G(g) &= \{g'(g) \mid g' \in G\} \\ &= \{g'gg'^{-1} \mid g' \in G\} \\ &= \{gg'g'^{-1} \mid g' \in G\} \qquad (g'g = gg' \text{ as } G \text{ is abelian}) \\ &= \{g \mid g' \in G\} = \{g\}\,. \end{aligned}$$

Conversely, suppose G acts on itself by conjugation and each conjugacy class is of size 1, then G must be abelian.

⟦Pf: Take g, $h \in G$, we have to prove $gh = hg$, i.e. $ghg^{-1} = h$.
But ghg^{-1} is in the orbit

$$G(h) = \{g'(h) \mid g' \in G\} = \{g'hg'^{-1} \mid g' \in G\}$$

of h, as in particular we can take $g' = g$.
By assumption, this orbit has a single element, and putting $g' = e$, we conclude that this element must be h, so ghg^{-1} and h have to agree.⟧

In summary, we get

Proposition. *Conjugacy classes of G are all of size 1 $\Leftrightarrow$ G is abelian.*

Examples (ctd).

(2) Consider the cyclic group of order $n \geq 1$ as a *subgroup of* $\mathbb{C}$:

$$\begin{aligned} C_n &= \{e^{2\pi ik/n} \mid k \in \mathbb{Z}\} \\ &= \{e^{2\pi ik/n} \mid k \in \mathbb{Z}\} \end{aligned}$$

C_n is abelian (as a subgroup of the group $(\mathbb{C}^*, \cdot)$, the units in the field (hence also ring) $\mathbb{C}$), and so its conjugacy classes are given by

$$\{e^0\}, \{e^{2\pi i/n}\}, \ldots, \{e^{2\pi i(n-1)/n}\}\,.$$

(3) We have seen already much earlier that the symmetric group S_3 has two non-trivial conjugacy classes, one consisting of the order 3 elements $\{(1\,2\,3), (3\,2\,1)\}$ and another one of the elements of order 2, i.e. by $\{(1\,2), (2\,3), (3\,1)\}$.

(4) The dihedral group

$$D_5 = \langle r, h \mid r^5 = e = h^2, hrh^{-1} = r^{-1} \rangle$$

has its elements listed as $\{r^j h^i \mid 0 \leq j \leq 4,\ 0 \leq i \leq 1\}$.
The conjugacy class of r^k in D_5 for any fixed k $(0 \leq k \leq 4)$ can be computed

as follows

$$\begin{aligned}
\mathrm{ccl}_{D_5}(r^k) &= \{(r^jh^i)r^k(r^jh^i)^{-1} \mid 0 \le j \le 4,\ 0 \le i \le 1\} \\
&= \{r^jh^ir^kh^{-i}r^{-j} \mid 0 \le j \le 4,\ 0 \le i \le 1\} \\
&= \underbrace{\{r^jr^kr^{-j} \mid 0 \le j \le 4\}}_{i=0} \cup \underbrace{\{r^jhr^kh^{-1}r^{-j} \mid 0 \le j \le 4\}}_{i=1} \\
&= \{r^k\} \cup \{r^j \underbrace{hr^kh^{-1}}_{r^{-k}} r^{-j} \mid 0 \le j \le 4\} \\
&= \{r^k\} \cup \{r^{-k}\}\,.
\end{aligned}$$

This latter set has two elements for $1 \le k \le 4$, and one element for $k = 0$.

Similarly, any other element in D_5 can be written as r^kh, with k fixed, and we find for the conjugacy class

$$\begin{aligned}
\mathrm{ccl}_{D_5}(r^kh) &= \{(r^jh^i)r^kh(r^jh^i)^{-1} \mid 0 \le j \le 4,\ 0 \le i \le 1\} \\
&= \{r^jh^ir^khh^{-i}r^{-j} \mid 0 \le j \le 4,\ 0 \le i \le 1\} \\
&= \underbrace{\{r^jr^khr^{-j} \mid 0 \le j \le 4\}}_{i=0} \cup \underbrace{\{r^jhr^kr^{-j} \mid 0 \le j \le 4\}}_{i=1} \\
&= \{r^jr^kr^jh \mid 0 \le j \le 4\} \cup \{r^jr^{j-k}h \mid 0 \le j \le 4\}
\end{aligned}$$

and both sets on the right hand side agree; they can be written as

$$\{r^ih \mid 0 \le i \le 4\}\,.$$

Summary: the conjugacy classes of D_5 are

$$\{e\}, \{r, r^{-1}\} = \{r^4, r^{-4}\}, \{r^2, r^{-2}\} = \{r^3, r^{-3}\}, \{h, rh, r^2h, r^3h, r^4h\}\,.$$

These are the orbits under conjugation.
The corresponding stabilisers are

$$\begin{aligned}
G_e &= \{g \in G \mid geg^{-1} = e\} = D_5\,, \\
G_r &= \langle r \rangle = G_{r^2} = G_{r^3} = G_{r^4} \qquad \text{(5 elements in each)} \\
G_{r^kh} &= \{e, r^kh\} \qquad \text{(2 elements in each).}
\end{aligned}$$

If we consider the size of the stabilisers in the above example and compare them with the size of the respective orbits, we are led to the following pairs $(\#G(e), \#G_e) = (1, 10)$, $(\#G(r), \#G_r) = (2, 5)$, $(\#G(rh), \#G_{rh}) = (5, 2)$, and in each case the two numbers multiply to 10. This is an illustration of a general phenomenon, which we are aiming at: the Orbit-Stabiliser Theorem. For this, recall the notion of equivalence relation on a set X: it is a *binary relation* $\sim$ on X (i.e. we attach a value [here Boolean, “true” or “false”] to each pair of elements in X), satisfying the following three conditions (R) “reflexivity”: $x \sim x$, (S) “symmetry”: if $x \sim y$ then $y \sim x$ and (T) “transitivity”: if $x \sim y$ and $y \sim z$ then $x \sim z$.

Now note that being in the same left coset with respect to a subgroup H in a group G defines an equivalence relation, and that the cosets w.r.t. H all have the same size. Hence we can formulate:

Orit-Stabiliser Theorem.

Orbit-Stabiliser Theorem. Suppose G acts on a set X. Then for any $x \in X$ there is a bijection

$$\begin{aligned} \beta : G(x) &\xrightarrow{1:1} \{\text{left cosets of } G_x \text{ in } G\} \\ g(x) &\mapsto gG_x\,. \end{aligned}$$

The proof becomes rather straightforward once we realise the following equivalence: for any g and $h \in G$

$$\begin{aligned} g(x) = h(x) \quad &\Leftrightarrow \quad g^{-1}g(x) = g^{-1}h(x) \qquad (\text{multiply on the left by } g^{-1}) \\ &\Leftrightarrow \quad x = g^{-1}h(x) \\ &\Leftrightarrow \quad g^{-1}h \in G_x \qquad (\text{by definition of stabiliser}) \\ &\Leftrightarrow \quad g^{-1}hG_x = G_x \qquad (\text{as } G_x \text{ is a sub}\textit{group}) \\ &\Leftrightarrow \quad hG_x = gG_x\,. \end{aligned}$$

Now we use the above equivalence to establish the following two statements.

(i) Well-definedness of β (simply use implication "$\Rightarrow$" from the above).

(ii) Injectivity of β (use implication "$\Leftarrow$" from the above).

It remains to verify surjectivity of the map given. So suppose that we are given a coset C, then we need to write it in the form $\tilde{g}G_x$ for some $\tilde{g}$ in G.
For $\tilde{g}$ we take any element of C (which is non-empty) and then show that $C = \tilde{g}G_x$: Clearly $\tilde{g} = \tilde{g}e$ lies in $\tilde{g}G_x$, and hence $C = \tilde{g}G_x$ ⟦cosets either are disjoint or agree⟧ Then the element $\tilde{g}(x)$ of $G(x)$ is indeed mapped under β to $\beta\big(\tilde{g}(x)\big) = \tilde{g}G_x = C$, establishing surjectivity of β.

We will often use the following important consequence of the Orbit-Stabiliser Theorem:

Corollary. If G is finite, acting on a finite set X, then for any $x \in X$ we have

$$|G(x)| \cdot |G_x| = |G|\,,$$

i.e. the size of its orbit $G(x)$ is "complementary" to the size of its stabiliser G_x.

Proof. Taking sizes in the statement of the Orbit-Stabiliser Theorem we have

$$|G(x)| = |\{\text{left cosets of } G_x \text{ in } G\}|\,. \qquad (*)$$

But all the cosets with respect to G_x have the same size, i.e.

$$|G_x| = |eG_x| = |gG_x| \qquad \text{for any } g \in G\,.$$

Hence $|G|/|G_x|$ is the number of cosets w.r.t. G_x in G, and by $(*)$ above we find indeed

$$|G(x)| = \frac{|G|}{|G_x|}\,,$$

and the claim follows. □

Remark. Note that the statement of the corollary still makes sense if the set X or the group G is infinite, by the usual rules of calculus of cardinal numbers, e.g. $\infty \cdot n = \infty \cdot \infty = \infty$ $(n > 0)$.

Corollary. If the finite group G acts on the finite set X, then the orbit lengths divide the group order, i.e.

$$|G(x)| \quad \text{divides} \quad |G| \quad \text{for any } x \in X\,.$$

In particular, the size of each conjugacy class in G divides $|G|$.

Example. The dihedral group D_n, for n *odd*, has orbits and stabilisers as follows:

Elements	e	$r \quad r^{-1}$	$r^2 \quad r^{-2}$	$\dots$	$r^{\frac{n-1}{2}} \quad r^{-\frac{n-1}{2}}$	$h\ rh\ \dots r^{n-1}h$
Orbits	$\{e\}$	$\{r, r^{-1}\}$	$\{r^2, r^{-2}\}$	$\dots$	$\{r^{\frac{n-1}{2}}, r^{-\frac{n-1}{2}}\}$	$\{h, rh, \dots, r^{n-1}h\}$
Orb. Sizes	1	2	2	$\dots$	2	n
Stabilisers	D_n	$\langle r\rangle$	$\langle r^2\rangle$	$\dots$	$\langle r^{\frac{n-1}{2}}\rangle$	$\langle h\rangle, \langle rh\rangle, \dots, \langle r^{n-1}h\rangle$
Stab. Sizes	$2n$	n	n	$\dots$	n	2

The stabilisers for elements of the *same* orbit are related in a simple way to each other.

Proposition. *Suppose x lies in the G-orbit of y; then G_x and G_y are conjugate to each other, i.e.*

$$G_x = hG_yh^{-1} \qquad \textit{for some } h \in G\,.$$

Proof. By assumption $x = h(y)$ for some $y \in G$. Now rewrite G_x in several steps:

$$\begin{aligned} G_x &= \{g \in G \mid g(x) = x\} \\ &= \{g \in G \mid g(h(y)) = h(y)\} \\ &= \{g \in G \mid h^{-1}(g(h(y))) = \underbrace{h^{-1}(h(y))}_{=y}\}\,. \end{aligned}$$

Now put $g' = h^{-1}gh$, so that $g = hg'h^{-1}$. Then the right hand side can be written

$$\begin{aligned} &= \{hg'h^{-1} \in G \mid g'(y) = y\} \\ &= h\{g' \in G \mid g'(y) = y\}h^{-1} \\ &= hG_yh^{-1}\,. \end{aligned}$$

6. First structural results (Cauchy's Theorem; groups of order $2p$)

We are now aiming at our first structural results on groups, using the notion of a group action. In one of the previous homeworks, we have seen that the converse to Lagrange's Theorem does not hold. Nevertheless, we get a "partial converse" in the following statement, due to Cauchy.

Cauchy's Theorem. Let G be a finite group and p a prime such that $p\big||G|$. Then there is a subgroup of G of order p.

Proof. For the proof, we want to find an element $x \in G$ such that $x^p = e$, $x \neq e$. The clever idea is to look at

$$\underbrace{G \times G \times \cdots \times G}_{p \text{ factors}} \qquad \Big[:= \Big(\big((G \times G) \times G\big) \times \dots\Big) \times G\Big]\,,$$

which forms a group itself. (Why?) Moreover, we look at the subset

$$\Omega := \{(x_1, x_2, \dots, x_p) \mid x_1x_2\cdots x_p = e\}\,.$$

There is an action of the group $\mathbb{Z}_p$ on $G \times G \times \cdots \times G$ by "cyclically shifting", i.e.

$$\overline{1} : (x_1, x_2, \dots, x_p) \quad \mapsto \quad (x_2, x_3, \dots, x_p, x_1)$$

and more generally

$$\overline{m} : (x_1, x_2, \dots, x_p) \quad \mapsto \quad (x_{m+1}, x_{m+2}, \dots, x_p, x_1, \dots, x_m)\,.$$

This action induces an action of $\mathbb{Z}_p$ *also on* Ω.
⟦If $(x_1, x_2, \dots, x_p) \in \Omega$ then $x_1 x_2 \cdots x_p = e$ but then also $x_2 \cdots x_p = x_1^{-1}$ and hence $x_2 \cdots x_p x_1 = e$, i.e. $(x_2, x_3, \dots, x_p, x_1) \in \Omega$.
Inductively, one shows that $(x_{m+1}, x_{m+2}, \dots, x_p, x_1, \dots, x_m) \in \Omega$ for any $m = 1, \dots, p$.⟧
Now we use that the order of any $\mathbb{Z}_p$-orbit in Ω divides the order of the group $\mathbb{Z}_p$ itself, i.e. divides p, so is either 1 or p.

There is one obvious orbit of size 1, given by

$$(e, e, \dots, e) \in \Omega \subset G \times \cdots \times G\,.$$

We will now establish that there must be another such size-1-orbit, and this will then provide an x with the desired properties (i.e. with $x^p = e$, $x \neq e$).

First we determine the size of Ω in relation to the size of G.

$$|\Omega| = |G|^{p-1}\,. \qquad (*)$$

⟦This holds simply because we can choose $x_1, \dots, x_{p-1}$ independently in G and then x_p is already determined by the condition $x_1 x_2 \cdots x_p = e$ (in fact, $x_p = (x_1 x_2 \cdots x_{p-1})^{-1}$).⟧
We know that Ω is partitioned into orbits under the $\mathbb{Z}_p$-action, and the corresponding orbits have size 1 or p (as they need to divide the order of the group that is acting), so we get a disjoint union of orbits

$$\Omega = \bigcup \{\text{orbits of size } 1\} \;\cup\; \bigcup \{\text{orbits of size } p\}\,.$$

Taking sizes, this becomes

$$|\Omega| = \sum_{\text{orbits of size } 1} 1 \;+ \sum_{\text{orbits of size } p} p\,,$$

and the left hand side is divisible by p by $(*)$. Hence p also divides the left term on the right hand side which counts the number of orbits of size 1 under the $\mathbb{Z}_p$-action. For this to be possible, there must be at least one (in fact $p-1$) such orbits of size 1 different from the one given above.
Any such orbit is necessarily of the form $\{(g, g, \dots, g)\}$ for some $g \in G$, $g \neq e$.
Now we are done, as such a g satisfies $(g, g, \dots, g) \in \Omega$, i.e. $\underbrace{g \cdot g \cdots g}_{p \text{ factors}} = e$. □.

As a nice application of Cauchy's Theorem, we get:

Theorem. *Any group G of order $2p$, where p is an odd prime, is either cyclic or dihedral.*

Proof. Cauchy's Theorem immediately gives us the existence of an element a of order 2 and an element b of order p. Putting $B = \langle b \rangle$, we see that B has order p and so G partitions into two cosets of order p.
In fact, we claim that aB is a coset different from B ⟦Clearly, any element in B

has odd order, while a is of order 2, so $a \notin B$ and hence $aB \neq B$.⟧
In order to check the dihedral relation which here amounts to $aba^{-1} = b^{-1}$ we try to find ba in any of the two cosets B and aB.
It cannot lie in the former, otherwise $ba = b^k$ for some $k \in \mathbb{Z}$, whence $a = b^{k-1} \in B$ which we already excluded.
Hence there must be a $k \in \{1, \ldots, p\}$ such that $ba = ab^k$. We now find the restrictions on k:

$$\begin{aligned} ba &= ab^k \\ \Rightarrow \ aba &= b^k \qquad \text{multiply by } a \text{ on left} \\ \Rightarrow \ b &= ab^k a \qquad \text{multiply by } a \text{ on right} \\ &= \underbrace{(aba) \cdots (aba)}_{k \text{ factors}} \\ &= b^k \qquad \text{as } b = ab^k a \text{ by the above} \\ &= (b^k)^k = b^{k^2} \end{aligned}$$

Hence (as b is of order p) we get for the exponents that $k^2 - 1 \equiv 0 \pmod{p}$, so p divides one of the factors $k-1$ or $k+1$, hence $k = 1$ or $k = p-1$.
In the first case, the group is cyclic, in the second case it is dihedral. □

Note: This result also holds for the prime $p = 2$ if we introduce D_2 as the group given by generators and relations D_n with formally putting $n = 2$. ⟦Some authors in fact do so.⟧
Now this D_2 happens to be isomorphic to V, the Klein 4-group ⟦try to establish the relations that hold for the elements in V from the ones for D_2, for example⟧, so is a bit different from the other dihedral groups in that it is commutative.

Why not celebrate our new structure theorem on groups with a limerick?

Twoxp

If a group has the size two times p
for that p a prime not less than 3,
apply ou-r sly trick
to show that it's cyclic
or dihedral; what else could it be?

H.G.

7. Conjugacy classes of S_n and A_n.

Recall that we had determined the conjugacy classes of S_3 which are given by

$$\{e\}, \ \{(1\,2), (2\,3), (3\,1)\} \ \text{ and } \{(1\,2\,3), (3\,2\,1)\}\,.$$

We can see that each element in a given conjugacy class here has the same "shape". This is an instance of a more general phenomenon:

Definition. Let $x \in S_n$, $x \neq e$, be written as a product of *disjoint* cycles, i.e.

$$x = (a_1\, a_2 \ldots a_{k_1})\,(b_1\, b_2 \ldots b_{k_2}) \ldots (t_1\, t_2\, \ldots t_{k_r})\,,$$

where $r \geq 1$, $k_1 \leq k_2 \leq \cdots \leq k_r$, and of course $n \geq k_1 + \cdots + k_r$.
Then we say that x has **cycle shape** $[k_1, k_2, \ldots, k_r]$.

Examples. 1. The cycle shape of $x = (1\,2)(3\,5\,7)(8\,9\,4)$ is $[2, 3, 3]$.
2. The cycle shape of $x = (1\,2)(3\,5\,7)(8\,9\,3)$ is *not* $[2, 3, 3]$, as it is not a product of disjoint cycles; instead we have $x = (1\,2)(8\,9\,3\,5\,7)$, and so it is of cycle shape $[2, 5]$.

Proposition. Let $(i_1\,i_2 \dots i_k)$ be a k-cycle in S_n $(n \geq k)$. Then for any $g \in S_n$ we can read off the action of g on x by conjugation as

$$gxg^{-1} = \big(g(i_1)\,g(i_2) \dots g(i_k)\big)\,,$$

where we view g as a permutation of $\{1, \dots, n\}$ on the RHS.

Example. Let $x = (2\,5\,4) \in S_5$ and let $g = (1\,2\,3\,5\,4)$. Then, as a permutation, g satisfies $g(2) = 3$, $g(5) = 4$ and $g(4) = 1$, so the proposition implies that

$$g\,x\,g^{-1} = \big(g(2)\,g(5)\,g(4)\big) = (3\,4\,1)\,.$$

Indeed, we can check in our usual cycle notation that $(1\,2\,3\,5\,4)(2\,5\,4)(4\,5\,3\,2\,1) = (1\,3\,4)$, which agrees with the above.

Proof. Write $T = \{i_1, \dots, i_k\}$ (i.e. the set of indices in the cycle form of x). We distinguish two cases.

Case 1: Let $j \in T$, then $j = i_r$ for some $r \in \{1, \dots, k\}$ and we find

$$gxg^{-1}\big(g(i_r)\big) = g\,\underbrace{x(i_r)}_{=i_{r+1}} = \begin{cases} g(i_{r+1}) & \text{if } 1 \leq r < k\,, \\ g(i_1) & \text{if } r = k\,. \end{cases}$$

Case 2: Let $j \notin T$, then gxg^{-1} leaves $g(j)$ fixed:

$$gxg^{-1}\big(g(j)\big) = g\,\underbrace{x(j)}_{=j} = g(j)\,.$$

Hence gxg^{-1} is the bijection of $\{1, \dots, n\}$ that is given in cycle form by $\big(g(i_1)\,g(i_2) \dots g(i_k)\big)$.

Putting this together for a product of disjoint cycles gives:

Theorem. For $x \in S_n$ the conjugacy class $\mathrm{ccl}_{S_n}(x)$ consists of all permutations which have the same cycle shape as x.

Proof. Let $x = (a_1\,a_2 \dots a_{k_1})\,(b_1\,b_2 \dots b_{k_2}) \dots (t_1\,t_2\, \dots t_{k_r})$ be the product of *disjoint* cycles of cycle shape $[k_1, k_2, \dots, k_r]$. Then

$$\begin{aligned} gxg^{-1} &= g(a_1\,a_2 \dots a_{k_1})\,(b_1\,b_2 \dots b_{k_2}) \dots (t_1\,t_2\, \dots t_{k_r})\,g^{-1} \\ &= \underbrace{g(a_1\,a_2 \dots a_{k_1})g^{-1}}_{=(g(a_1)\,g(a_2) \dots g(a_{k_1}))}\;\underbrace{g(b_1\,b_2 \dots b_{k_2})g^{-1}}_{=(g(b_1)\,g(b_2) \dots g(b_{k_2}))}\;\dots\;\underbrace{g(t_1\,t_2\, \dots t_{k_r})g^{-1}}_{=(g(t_1)\,g(t_2) \dots g(t_{k_r}))} \\ &= (g(a_1)\,g(a_2) \dots g(a_{k_1}))\;(g(b_1)\,g(b_2) \dots g(b_{k_2}))\;\dots\;(g(t_1)\,g(t_2) \dots g(t_{k_r}))\,, \end{aligned}$$

which has the *same* cycle shape as x ⟦note that all cycles on the RHS are disjoint as g is a bijection of $\{1, \dots, n\}$.⟧

On the other hand, given x and y of the same cycle shape,

$$\begin{aligned} x &= (a_1\,a_2 \dots a_{k_1})\,(b_1\,b_2 \dots b_{k_2}) \dots (t_1\,t_2\, \dots t_{k_r})\,, \\ y &= (a'_1\,a'_2 \dots a'_{k_1})\,(b'_1\,b'_2 \dots b'_{k_2}) \dots (t'_1\,t'_2\, \dots t'_{k_r})\,, \end{aligned}$$

there is a bijection of $\{1, \dots, n\}$ that sends $a_1 \mapsto a'_1, \dots t_{k_r} \mapsto t'_{k_r}$ since all the indices in the above product of cycles for x are mutually different (as well as for

y). Hence we can view such a bijection (which in general is not unique) as an element $g \in S_n$ and we have $gxg^{-1} = y$ (where we use the above proposition for each of the cycles involved).

Examples. 1. The conjugacy classes in S_4 are given by

$$\begin{aligned} &\{e\}, \quad \{(1\,2),(1\,3),(1\,4),(2\,3),(2\,4),(3\,4)\}, \\ &\{(1\,2\,3),(3\,2\,1),(2\,3\,4),(4\,3\,2),(3\,4\,1),(1\,4\,3),(4\,1\,2),(2\,1\,4)\}, \\ &\{(1\,2\,3\,4),(1\,2\,4\,3),(1\,3\,4\,2),(1\,3\,2\,4),(1\,4\,2\,3),(1\,4\,3\,2)\}, \\ &\{(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3)\}\,. \end{aligned}$$

Their cycle shapes are $[1]$, $[2]$, $[3]$, $[4]$, $[2,2]\}$, respectively.

2. In S_5 we get the same cycle shapes as for S_4 , together with two new ones: $[5]$ and $[2,3]$.

3. In S_6 we get the same cycle shapes as for S_5 , together with four new ones: $[6]$, $[4,2]$, $[3,3]$ and $[2,2,2]$.

For general n , we can enumerate the cycle shapes simply by running through all non-decreasing partitions of n and dropping the “1”s (except in the degenerate case $[1]$). So this gives us a complete list of conjugacy classes.

How many elements are there in a given conjugacy class of S_n ?

Claim 1: For an m-cycle $x = (a_1 \dots a_m) \in S_n$ we get

$$|\mathrm{ccl}_{S_n}(x)| = \frac{n(n-1)\dots(n-m+1)}{m}\,.$$

⟦Pf: Counting all the possibilities for writing $(a_1 \dots a_m)$ with different $a_i \in \{1,\dots,n\}$, we find $n(n-1)\dots(n-m+1)$ [we have n choices for the first entry, then only $n-1$ choices left for the second entry, etc.].

Then we realise that we overcounted by a factor of m since there are precisely m ways to write a given m-cycle.⟧

Claim 2: If $x \in S_n$ is of cycle shape $[m_1,\dots,m_r]$ with $m_1 < m_2 < \dots < m_r$ (in particular all the m_i are mutually different) then the number of elements for that cycle shape is given by

$$\begin{aligned} \gamma(n;m_1,\dots,m_r) := \tfrac{n(n-1)\dots(n-m_1+1)}{m_1} \cdot \tfrac{(n-m_1)(n-m_1-1)\dots(n-m_1-m_2+1)}{m_2} \cdots \\ \cdots \tfrac{\left(n-\sum_{i=1}^{r-1} m_i\right)\left(n-(\sum_{i=1}^{r-1} m_i)-1\right)\dots\left(n-(\sum_{i=1}^{r} m_i)+1\right)}{m_r}\,. \end{aligned}$$

⟦The proof is essentially the same as for Claim 1, together with induction on r .⟧

Claim 3: If $x \in S_n$ is of general cycle shape $[\underbrace{m_1,\dots,m_1}_{s_1},\underbrace{m_2,\dots,m_2}_{s_2},\dots,\underbrace{m_r,\dots,m_r}_{s_r}]$, still with $m_1 < m_2 < \dots < m_r$ (and $s_1,\dots,s_r \geq 1$), then the number of elements for that cycle shape is given by

$$\frac{\gamma(n;\underbrace{m_1,\dots,m_1}_{s_1},\underbrace{m_2,\dots,m_2}_{s_2},\dots,\underbrace{m_r,\dots,m_r}_{s_r})}{s_1!\,s_2!\cdots s_r!}\,.$$

The reason for these factorial terms comes from the fact that disjoint cycles commute, so if there are, e.g., s_1 cycles of length m_1 we have overcounted by a factor of $s_1!$ since we can arbitrarily permute these cycles without changing the cycle shape.

Example. (Conjugacy classes for S_4.) A conjugacy class consists of all elements of a given cycle shape, hence we find the sizes of different conjugacy classes by enumerating all the elements of a given cycle shape.
For S_4 we get the following table.

Cycle shapes of S_4	$[1]$	$[2]$	$[3]$	$[4]$	$[2,2]$
Sizes	1	$\frac{4\cdot 3}{2}=6$	$\frac{4\cdot 3\cdot 2}{3}=8$	$\frac{4\cdot 3\cdot 2\cdot 1}{4}=6$	$\frac{\frac{4\cdot 3}{2}\frac{2\cdot 1}{2}}{2}=3$

Normal subgroups of S_n and A_n.

As an application of the determination of cycle shapes (and their orders) for S_n and for A_n we can sometimes easily determine all their normal subgroups. For this, we recall a previous characterization of normal subgroups.

Proposition. *Let H be a subgroup of G. Then we have*

$$H \text{ is normal in } G \quad \Leftrightarrow \quad H \text{ is a union of conjugacy classes of } G\,.$$

But we should keep in mind the following

Note. Suppose there is a sum of conjugacy class order which divides the group order. Then this is in general *not* sufficient for a normal subgroup to exist!

Find all the normal subgroups of S_4: from the above table we get all the conjugacy classes for S_4.

By the above proposition, a normal subgroup N of S_4 is the union of conjugacy classes, hence its size is a sum of the sizes 1, 6, 8, 6 and 3, i.e. $|N| = \varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 6 + \varepsilon_3 \cdot 8 + \varepsilon_4 \cdot 6 + \varepsilon_5 \cdot 3$, with $\varepsilon_j \in \{0,1\}$ $(j = 1, \dots, 5)$.

Clearly, ε_1 must be 1, as the identity element must lie in any subgroup. By Lagrange, the sizes of contributing conjugacy classes must add up to a divisor of $|G| = 24$.

The only such possibilities are $1 + 3$ and $1 + 3 + 8$.

In the first case, we get $\mathrm{ccl}_{S_4}\big((1)\big) \cup \mathrm{ccl}_{S_4}\big((1\,2)(3\,4)\big)$, which indeed form a group, the Klein 4-group. Note that we need to check closure under composition.

In the second case, we find $\mathrm{ccl}_{S_4}\big((1)\big) \cup \mathrm{ccl}_{S_4}\big((1\,2)(3\,4)\big) \cup \mathrm{ccl}_{S_4}\big((1\,2\,3)\big)$; but these are precisely the 12 even permutations in S_4 which we already know to form a subgroup, denoted A_4.

In summary, we get that there are two non-trivial normal subgroups for S_4 (the trivial subgroups being $\{e\}$ and S_4 itself).