Basics on set-theoretic properties of functions

We assume familiarity with sets and functions. The following lists some properties of images and pre-images of functions which are constantly used in Topology III. The reader should check the validity of these statements, none of which is very difficult. We begin with some notation.

Let X, Y be sets, and $f : X \to Y$ a function. Let $A \subset X$. The *image* of A under f is denoted by $f(A)$, and is a subset of Y defined as

$$f(A) \quad = \quad \{y \in Y \,|\, y = f(a) \text{ for some } a \in A\}.$$

Let $B \subset Y$. The *pre-image*, or *inverse image*, of B under f is denoted by $f^{-1}(B)$, and is a subset of X defined as

$$f^{-1}(B) \quad = \quad \{x \in X \,|\, f(x) \in B\}.$$

Note that if $f : X \to Y$ is bijective, we denote the inverse function as $f^{-1} : Y \to X$. Then $f^{-1}(B)$ can be both image of f^{-1} or pre-image of f. However, both are the same in this instance, so there is no ambiguity.

The pre-image behaves much nicer with respect to set-theoretic operations. In the following, $f : X \to Y$ is a function, $B_0, B_1 \subset Y$.

1. $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
2. $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.
3. $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.
4. $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.

Here we write $-$ for the set-theoretic difference, that is, $B_0 - B_1 = \{b \in B_0 \,|\, b \notin B_1\}$. We do not need that $B_1 \subset B_0$, but we get $(B_0 - B_1) \subset B_0$.

Properties 2. and 3. also hold for arbitrary unions and intersections.

For the image, we get the following, where $A_0, A_1 \subset X$.

1. $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.
2. $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.
3. $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.
4. $f(A_0 - A_1) \supset f(A_0) - f(A_1)$.

Again, properties 2. and 3. hold for arbitrary unions and intersections. Also, there exist examples where equality in 3. and 4. fail.

Let $A \subset X$ and $B \subset Y$. Then

1. $f^{-1}(f(A)) \supset A$, and equality holds if f is injective.
2. $f(f^{-1}(B)) \subset B$, and equality holds if f is surjective.

Let $f : X \to Y$ and $g : Y \to Z$ be functions, $A \subset X$ and $C \subset Z$. Then

1. $(g \circ f)(A) = g(f(A))$.
2. $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.