The Walker conjecture for chains in $\mathbb{R}^d$

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Abstract

A chain is a configuration in $\mathbb{R}^d$ of segments of length $\ell_1, \dots \ell_{n-1}$ consecutively joined to each other such that the resulting broken line connects two given points at a distance ℓ_n. For fixed generic set of length parameters the space of all chains in $\mathbb{R}^d$ is a closed smooth manifold of dimension $(n-2)(d-1)-1$. In this paper we study cohomology algebras of spaces of chains. We give a complete classification of these spaces (up to equivariant diffeomorphism) in terms of linear inequalities of a special kind which are satisfied by the length parameters $\ell_1, \dots, \ell_n$. This result is analogous to the conjecture of K. Walker which concerns the special case $d = 2$.

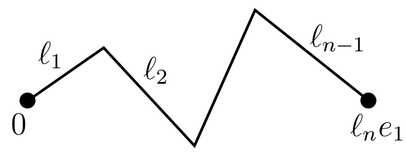
MSC classes: 55R80, 70G40, 57R19

Introduction

For $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{R}^n_{>0}$ and d a positive integer, define the subspace $\mathcal{C}^n_d(\ell)$ of $(S^{d-1})^{n-1}$ by

$$\mathcal{C}^n_d(\ell) = \{z = (z_1, \dots, z_{n-1}) \in (S^{d-1})^{n-1} \mid \sum_{i=1}^{n-1} \ell_i z_i = \ell_n \, e_1\} ,$$

where $e_1 = (1, 0, \dots, 0)$ is the first vector of the standard basis $e_1, \dots, e_d$ of $\mathbb{R}^d$. An element of $\mathcal{C}^n_d(\ell)$, called a *chain*, can be visualised as a configuration of $(n-1)$-segments in $\mathbb{R}^d$, of length $\ell_1, \dots, \ell_{n-1}$, joining the origin to $\ell_n e_1$. The vector ℓ is called the *length vector*.



The group $O(d-1)$, viewed as the subgroup of $O(d)$ stabilising the first axis, acts naturally (on the left) upon $\mathcal{C}_d^n(\ell)$. The quotient $SO(d-1)\backslash\mathcal{C}_d^n(\ell)$ is the *polygon space* $\mathcal{N}_d^n$, usually defined as

$$\mathcal{N}_d^n(\ell) = SO(d)\Big\backslash \left\{ z \in (S^{d-1})^n \,\middle|\, \sum_{i=1}^n \ell_i z_i = 0 \right\}.$$

When $d = 2$ the space of chains $\mathcal{C}_2^n(\ell)$ coincides with the polygon space $\mathcal{N}_2^n(\ell)$. Descriptions of several chain and polygon spaces are provided in [8] (see also [7] for a classification of $\mathcal{C}_d^4(\ell)$).

A length vector $\ell \in \mathbb{R}_{>0}^n$ is *generic* if $\mathcal{C}_1^n(\ell) = \emptyset$, that is to say there is no collinear chain. It is proven in e.g. [7] that, for ℓ generic, $\mathcal{C}_d^n(\ell)$ is a smooth closed manifold of dimension

$$\dim \mathcal{C}_d^n(\ell) = (n-2)(d-1) - 1\,.$$

Another known fact is that if $\ell, \ell' \in \mathbb{R}_{>0}^n$ satisfy

$$(\ell'_1, \ldots, \ell'_{n-1}, \ell'_n) = (\ell_{\sigma(1)}, \ldots, \ell_{\sigma(n-1)}, \ell_n)$$

for some permutation σ of $\{1, \ldots, n-1\}$, then $\mathcal{C}_d^n(\ell')$ and $\mathcal{C}_d^n(\ell)$ are $O(d-1)$-equivariantly diffeomorphic, see [8, 1.5].

A length vector $\ell \in \mathbb{R}_{>0}^n$ is *ordered* if $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_n$.

A length vector $\ell \in \mathbb{R}_{>0}^n$ is *dominated* if $\ell_i \leq \ell_n$ for all $i = 1, \ldots, n-1$.

The goal of this paper is to show that for $d \geq 3$ the diffeomorphism types of spaces $\mathcal{C}_d^n(\ell)$ (for ℓ generic and dominated) are in one-to-one correspondence with some pure combinatorial objects, described below.

Theorem A. *Let $d \in \mathbb{N}$, $d \geq 3$. Then, the following properties of generic and dominated length vectors $\ell, \ell' \in \mathbb{R}_{>0}^n$ are equivalent:*

(a) $\mathcal{C}_d^n(\ell)$ and $\mathcal{C}_d^n(\ell')$ are $O(d-1)$-equivariantly diffeomorphic.

(b) $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z})$ and $H^*(\mathcal{C}_d^n(\ell'); \mathbb{Z})$ are isomorphic as graded rings.

(c) $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z}_2)$ and $H^*(\mathcal{C}_d^n(\ell'); \mathbb{Z}_2)$ are isomorphic as graded rings.

Moreover, if the vectors ℓ and ℓ' are ordered[1]*, then the above conditions are equivalent to:*

(d) *For a subset $J \subset \{1, \dots, n\}$ the inequality*

$$\sum_{i \in J} \ell_i < \sum_{i \notin J} \ell_i$$

holds if and only if

$$\sum_{i \in J} \ell'_i < \sum_{i \notin J} \ell'_i.$$

The equivalence (a) $\sim$ (d) means that the topology of the chain space $\mathcal{C}_d^n(\ell)$ determines the length vector ℓ, up to certain combinatorial equivalence.

In the case $d = 2$ we do not know if (c) $\Rightarrow$ (a) although the equivalences (a) $\sim$ (b) $\sim$ (d) are true. This is related to a conjecture of K. Walker [13] who suggested that planar polygon spaces are determined by their integral cohomology rings. The conjecture was proven for a large class of length vectors in [4] and the (difficult) remaining cases were settled in [11]. The spatial polygon spaces $\mathcal{N}_3^n$ are also determined up to diffeomorphism by their mod2-cohomology ring if $n > 4$, see [4, Theorem 3]. No such result is known for $\mathcal{N}_d^n$ when $d > 3$.

One may interpret Theorem A as follows. Consider the simplex $A^{n-1} \subset \mathbb{R}^n$ of dimension $n - 1$ given by the inequalities

$$0 < \ell_1 < \ldots < \ell_{n-1} < \ell_n = 1$$

and the hyperplanes $H_J \subset \mathbb{R}^n$ defined by the equations

$$\sum_{i \in J} \ell_i = \sum_{i \notin J} \ell_i,$$

for all possible subsets $J \subset \{1, \dots, n\}$. The connected components of the complement $A^{n-1} - \cup_J H_J$ are called *chambers*. Theorem A implies that for a fixed $d \geq 3$ the manifolds $\mathcal{C}_d^n(\ell)$ and $\mathcal{C}_d^n(\ell')$, where $\ell, \ell' \in (A^{n-1} - \cup_J H_J)$, are equivariantly diffeomorphic if and only if the vectors ℓ and ℓ' lie in the same chamber. Thus we obtain a one-to-one correspondence between the

[1] This can be achieved by a permutation of $\ell_1, \dots, \ell_{n-1}$, see above.

chambers and the equivariant diffeomorphism types of the manifolds $\mathcal{C}_d^n(\ell)$ for generic length vectors $\ell \in A^{n-1}$.

The number c_n of chambers in A^{n-1} grows fast with the number of parameters n. It was established in [10] that $c_3 = 2$, $c_4 = 3$, $c_5 = 7$, $c_6 = 21$, $c_7 = 135$, $c_8 = 2470$ and $c_9 = 175428$.

We now give the scheme of the proof of Theorem A. We first recall that the $O(d-1)$-diffeomorphism type of $\mathcal{C}_d^n(\ell)$ is determined by d and the sets of ℓ-short (or long) subsets, which play an important role all along this paper. A subset J of $\{1, \dots, n\}$ is *ℓ-short*, or just *short*, if

$$\sum_{i \in J} \ell_j < \sum_{i \notin J} \ell_j \,.$$

The reverse inequality defines *long* (or *ℓ-long*) subsets. Observe that ℓ is generic if and only if any subset of $\{1, \dots, n\}$ is either short or long.

The family of subsets of $\{1, \dots, n\}$ which are long is denoted by $\mathcal{L} = \mathcal{L}(\ell)$. Short subsets form a poset under inclusion, which we denote by $\mathcal{S} = \mathcal{S}(\ell)$. We are interested in the subposet

$$\dot{\mathcal{S}} = \dot{\mathcal{S}}(\ell) = \{J \subset \{1, \dots, n-1\} \mid J \cup \{n\} \in \mathcal{S}\} \,. \tag{1}$$

The following lemma is proven in [8, Lemma 1.2] (this reference uses the poset $\mathcal{S}_n(\ell) = \{J \in \mathcal{S} \mid n \in J\}$ which is determined by $\dot{\mathcal{S}}(\ell)$).

Lemma 0.1. *Let $\ell, \ell' \in \mathbb{R}^n_{>0}$ be generic length vectors. Suppose that $\dot{\mathcal{S}}(\ell)$ and $\dot{\mathcal{S}}(\ell')$ are isomorphic as simplicial complexes. Then $\mathcal{C}_d^m(\ell)$ and $\mathcal{C}_d^m(\ell')$ are $O(d-1)$-equivariantly diffeomorphic.* □

Lemma 0.1 gives the implication (d) ⇒ (a) in Theorem A.

Note that $H^*(\mathcal{C}_d^n; \mathbb{Z}_2) = 0$ if and only if $\mathcal{C}_d^n = \emptyset$, which happens if and only if $\{n\}$ is long. We can thus suppose that $\{n\}$ is short and hence $\dot{\mathcal{S}}(\ell)$ is determined by its subposet

$$\tilde{\mathcal{S}} = \tilde{\mathcal{S}}(\ell) = \dot{\mathcal{S}}(\ell) - \{\emptyset\} \,. \tag{2}$$

The poset $\tilde{\mathcal{S}}$ is an abstract simplicial complex (as a subset of a short subset is short) with vertex set contained in $\{1, \dots, n-1\}$. To prove Theorem A, it then suffices to show that the graded ring $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z}_2)$ determines $\tilde{\mathcal{S}}(\ell)$ when ℓ is dominated.

For a finite simplicial complex Δ whose vertex set $V(\Delta)$ is contained in $\{1,\dots,n\}$, consider the graded ring

$$\Lambda(\Delta) = \mathbb{Z}_2[X_1,\dots,X_n]/\mathcal{I}(\Delta)\,,$$

where $\mathcal{I}(\Delta)$ is the ideal of the polynomial ring $\mathbb{Z}_2[X_1,\dots,X_n]$ generated by X_i^2 and the monomials $X_{j_1}\cdots X_{j_k}$ when $\{j_1,\dots,j_k\}$ is not a simplex of Δ. Let Δ and Δ' be two finite simplicial complexes with vertex sets contained in $\{1,\dots,n\}$. By a result of J. Gubeladze, any graded ring isomorphism $\Lambda(\Delta)\approx\Lambda(\Delta')$ is induced by a simplicial isomorphism $\Delta\approx\Delta'$ (see [6, Example 3.6]; for more details, see [4, Theorem 8]). Hence, the implication (c) $\Rightarrow$ (d) of Theorem A will be established if we prove the following result:

Theorem B. *Let $\ell\in\mathbb{R}^n_{>0}$ be a generic dominated length vector. When $d\geq 3$, the subring $H^{(d-1)*}(\mathcal{C}^n_d(\ell);\mathbb{Z}_2)$ of $H^*(\mathcal{C}^n_d(\ell);\mathbb{Z}_2)$ is isomorphic to $\Lambda(\tilde{\mathcal{S}}(\ell))$.*

The implication (b) $\Rightarrow$ (c) follows since under the condition that ℓ is dominated, $H^*(\mathcal{C}^n_d(\ell);\mathbb{Z})$ is torsion free, see Theorem 2.1. Note also Remark 2.2 which shows that the condition that ℓ is dominated is necessary.

The proof of Theorem B is given in Section 4. The preceding sections are preliminaries for this goal. For instance, the computation of $H^*(\mathcal{C}^n_d(\ell);\mathbb{Z})$ as a graded abelian group, is given in Theorem 2.1.

1 Robot arms in $\mathbb{R}^d$

Let

$$\mathbb{S}=\mathbb{S}^n_d=\{\rho\colon\{1,\dots,n\}\to S^{d-1}\}\approx(S^{d-1})^n\,.$$

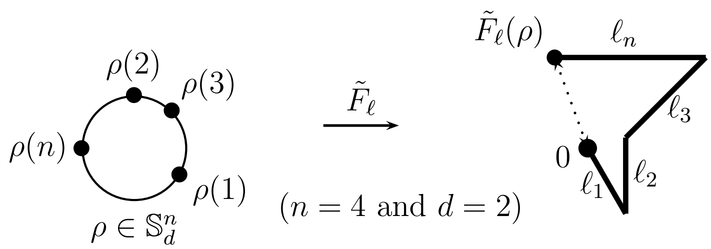
By post-composition, the orthogonal group $O(d)$ acts smoothly on the left upon $\mathbb{S}$. In [5, § 4–5], the quotient $W=SO(2)\backslash\mathbb{S}^n_2\approx(S^1)^{n-1}$ is used to get cohomological informations about $\mathcal{C}^n_2$. In this section, we extend these results for $d>2$. The quotient $SO(d)\backslash\mathbb{S}^n_d$ is no longer a convenient object to work with, so we replace it by the fundamental domain for the $O(d)$-action given by the submanifold

$$Z=Z^n_d=\{\rho\in\mathbb{S}\mid\rho(n)=-e_1\}\approx(S^{d-1})^{n-1}\,.$$

Observe that Z inherits an action of $O(d-1)$.

For a length vector $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n_{>0}$ the *ℓ-robot arm* is the smooth map $\tilde{F}_\ell\colon \mathbb{S} \to \mathbb{R}^d$ defined by

$$\tilde{F}_\ell(\rho) = \sum_{i=1}^{n} \ell_i \rho(i)\,.$$



Observe that the point $\rho \in \mathcal{C}^4_d$ in the above figure lies in Z. We also define an $O(d)$-invariant smooth map $\tilde{f}_\ell\colon \mathbb{S} \to \mathbb{R}$ by

$$\tilde{f}_\ell(\rho) = -|F_\ell(\rho)|^2\,.$$

The restrictions of $\tilde{F}$ and $\tilde{f}$ to Z are denoted by F and f respectively. Observe that

$$\mathcal{C} = \mathcal{C}^n_d(\ell) = f^{-1}(0) \subset Z\,.$$

Define

$$\mathbb{S}' = \mathbb{S} - \mathcal{C} \quad \text{and} \quad Z' = Z - \mathcal{C}$$

The restriction of $\tilde{f}$ and f to $\mathbb{S}'$ and Z' are denoted by $\tilde{f}'$ and f' respectively.

Denote by $Crit(g)$ be the subspace of critical points of a real value map g. One has $Crit(\tilde{f}) = \mathcal{C} \,\dot{\cup}\, Crit(\tilde{f}')$ and $Crit(f) = \mathcal{C} \,\dot{\cup}\, Crit(f')$, where $\dot{\cup}$ denotes the disjoint union. It is easy and well known that $\rho \in Crit(\tilde{f}')$ if and only if ρ is a collinear configuration, i.e. $\rho(i) = \pm\rho(j)$ for all $i, j \in \{1, \ldots, n\}$.

We will index the critical points of $\tilde{f}'$ and f' by the long subsets. For each $J \in \mathcal{L}$, let $Crit_J(\tilde{f}') \subset Crit(\tilde{f}')$ be defined by

$$Crit_J(\tilde{f}') = \{\rho \in \mathbb{S} \mid \kappa_J(j)\rho(j) = \kappa_J(i)\rho(i) \ \text{ for all } i, j \in \{1, \ldots, n\}\}\,.$$

where $\kappa_J\colon \{1, \ldots, n\} \to \{\pm 1\}$ the multiplicative characteristic function of J, defined by:

$$\kappa_J(i) = \begin{cases} -1 & \text{if } i \in J \\ 1 & \text{if } i \notin J\,. \end{cases}$$

In particular, $\kappa_{\bar{J}} = -\kappa_J$ if $\bar{J}$ is the complement of J in $\{1,\dots,n\}$. In words, $Crit_J(\tilde{f}')$ is the space of collinear configurations ρ which take constant values on J and $\bar{J}$ and such that $\rho(J) = -\rho(\bar{J})$. The space $Crit_J(\tilde{f}')$ is a submanifold of $\mathbb{S}$ diffeomorphic, via $\tilde{F}$, to the sphere in $\mathbb{R}^d$ of radius $\sum_{j\in J}\ell_j - \sum_{j\notin J}\ell_j$ (this number is positive since J is long). One has

$$Crit(\tilde{f}') = \dot{\bigcup_{J\in\mathcal{L}}} Crit_J(\tilde{f}')\,.$$

The $O(d)$-invariance of $\tilde{f}'$ has two consequences: each sphere $Crit_J(\tilde{f}')$ intersects Z transversally in the single point ρ_J and $Crit(f') = Crit(\tilde{f}') \cap Z$. Hence

$$Crit(f') = \{\rho_J \mid J \in \mathcal{L}\} \tag{3}$$

(note that $\rho_J \notin \mathcal{C}$ as ℓ is generic). As $\rho(n) = -e_1$ if $\rho \in Z$, the critical points ρ_J are of two types, depending on $n \in J$ or not:

$$\rho_J(i) = \begin{cases} \kappa_J(i)\,e_1 & \text{if } n \in J \\ -\kappa_J(i)\,e_1 & \text{if } n \notin J\,. \end{cases} \tag{4}$$

Lemma 1.1. *The map $f'\colon Z' \to (-\infty, 0)$ is a proper Morse function with set of critical points $\{\rho_J \mid J \in \mathcal{L}\}$. The index of ρ_J is $(d-1)(n-|J|)$.*

Proof. Because f' extends to $f\colon (Z,\mathcal{C}) \to ((-\infty,0],0)$, the map f' is proper. We already described $Crit(f')$ in (3). The non-degeneracy of ρ_J and the computation of its index are classical in topological robotics using arguments as in [7, proof of Theorem 3.2]. □

Consider the axial involution $\hat{\tau}$ on $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ defined by $\hat{\tau}(x,y) = (x,-y)$. It induces an involution τ on $\mathbb{S}$ and on Z. The maps $\tilde{f}$ and f are τ-invariant. Moreover, the critical set of $f'\colon Z' \to (-\infty,0)$ coincides with the fixed point set Z^τ. By Lemma 1.1 and [5, Theorem 4], this proves the following

Lemma 1.2. *The Morse function $f'\colon Z' \to (-\infty,0)$ is $\mathbb{Z}$-perfect, in the sense that $H_i(Z')$ is free abelian of rank equal to the number of critical points of index i. Moreover, the induced map $\tau_*\colon H_i(Z') \to H_i(Z')$ is multiplication by $(-1)^i$.* □

(Theorem 4 of [5] is stated for a Morse function $f\colon M \to \mathbb{R}$ where M is a compact manifold with boundary. To use it in the proof of Lemma 1.2, just replace Z' by $Z - N$ where N is a small open tubular neighbourhood of $\mathcal{C}$.)

We now represent a homology basis for Z and Z' by convenient closed manifolds. For $J \subset \{1, \dots, n\}$, define

$$\mathbb{S}_J = \{\rho \in \mathbb{S} \mid |\rho(J)| = 1\}$$

(the condition $|\rho(J)| = 1$ is another way to say that ρ is constant on J). The space $\mathbb{S}_J$ is a closed submanifold of $\mathbb{S}$ diffeomorphic to $(S^{d-1})^{n-|J|+1}$. As it is $O(d)$-invariant, it intersects Z transversally. Let

$$W_J = \mathbb{S}_J \cap Z \approx (S^{d-1})^{n-|J|} .$$

The manifold W_J is $O(d-1)$-invariant and then is τ-invariant. As in Formula (4), the dichotomy "$n \in J$ or not" occurs:

$$W_J = \begin{cases} \{\rho \in Z \mid \rho(J) = -e_1\} & \text{if } n \in J \\ \{\rho \in Z \mid |\rho(J)| = 1\} & \text{if } n \notin J\,. \end{cases} \tag{5}$$

We denote by $[W_J] \in H_{(d-1)(n-|J|)}(Z;\mathbb{Z})$ the class represented by W_J (for some chosen orientation of W_J). If J is long, then $W_J \subset Z'$ and we also denote by $[W_J]$ the class represented by W_J in $H_{(d-1)(n-|J|)}(Z';\mathbb{Z})$.

Lemma 1.3. *(a) $H_*(Z';\mathbb{Z})$ is freely generated by the classes $[W_J]$ for $J \in \mathcal{L}$.*

(b) $H_*(Z;\mathbb{Z})$ is freely generated by the classes $[W_J]$ for all $J \in \{1, \dots, n\}$ with $n \in J$.

Proof. For (a), we invoke [5, Theorem 5]. Indeed, the the collection of τ-invariant manifolds $\{W_J \mid J \in \mathcal{L}\}$ satisfies all the hypotheses of this theorem (see also [5, Lemma 8]).

Let $K = \{1, \dots, n-1\}$. The restriction of $\rho \in Z$ to K gives a diffeomorphism from $h\colon Z \xrightarrow{\approx} \mathbb{S}_K \approx (S^{d-1})^{n-1}$. By the Künneth formula, $H_*(\mathbb{S}_K;\mathbb{Z})$ is freely generated by the classes $[W_I]$ for all $I \subset K$. If $n \in J$, $h(W_J) = W_{J-\{n\}}$, which proves (b). □

Let $J, J' \subset \{1, \dots, n\}$. When $|J| + |J'| = n + 1$, one has $\dim W_J + \dim W_{J'} = \dim Z = \dim Z'$ and the intersection number $[W_J] \cdot [W_{J'}] \in \mathbb{Z}$ is defined (intersection in Z). We shall use the following formulae.

Lemma 1.4. *$J, J' \subset \{1, \ldots, n\}$ with $|J| + |J'| = n + 1$. Then*

$$[W_J] \cdot [W_{J'}] = \begin{cases} \pm 1 & \text{if } |J \cap J'| = 1 \\ 0 & \text{if } |J \cap J'| > 1 \text{ and } n \in J \cup J' \,. \end{cases}$$

Proof. Suppose that $J \cap J' = \{q\}$. Then $|J \cup J'| = |J| + |J'| - |J \cap J'| = n$. Then, $n \in J \cup J'$ and $W_J \cap W_{J'}$ consists of the single point $\rho_{J \cup J'}$ (satisfying $\rho_{J \cup J'}(i) = -e_1$ for all $i \in \{1, \ldots, n\}$). It is not hard to check that the intersection is transversal (see [5, proof of (34)]), so $[W_J] \cdot [W_{J'}] = \pm 1$.

In the case $|J \cap J'| > 1$, there exists $q \in J \cap J'$ with $q \neq n$. Let α be a rotation of $\mathbb{R}^d$ such that $\alpha(e_1) \neq e_1$. Let $h\colon Z \to Z$ be the diffeomorphism such that $h(\rho)(k) = \rho(k)$ if $k \neq q$ and $h(\rho)(q) = \alpha \circ \rho(q)$. We now use that $n \in J \cup J'$, say $n \in J'$. Then, $\rho(q) = -e_1$ for $\rho \in W_{J'}$. Hence, $h(W_J) \cap W_{J'} = \emptyset$. As h is isotopic to the identity of Z, this implies that $[W_J] \cdot [W_{J'}] = 0$. □

Remark 1.5. In Lemma 1.4, the hypothesis $n \in J \cup J'$ is not necessary if d is even, by the above proof, since there exists a diffeomorphism of S^{d-1} isotopic to the identity and without fixed point. But, for example, if $n = d = 3$, one checks that $[W_J] \cdot [W_{J'}] = \pm 2$ for $J = J' = \{1, 2\}$.

In the case $n \in J \cap J'$ and $|J| + |J'| = n+1$, Lemma 1.4 takes the following form:

$$[W_J] \cdot [W_{J'}] = \begin{cases} \pm 1 & \text{if } J \cap J' = \{n\} \\ 0 & \text{otherwise}\,. \end{cases} \tag{6}$$

Therefore, the basis $\{[W_J] \mid |J| = n - k, n \in J\}$ of $H_{k(d-1)}(Z; \mathbb{Z})$ has a dual basis (up to sign) $\{[W_J]^\sharp \in H_{(n-k)(d-1)}(Z; \mathbb{Z}) \mid |J| = n - k, n \in J\}$ for the intersection form, defined by $[W_J]^\sharp = [W_K]$, where $K = \bar{J} \cup \{n\}$.

We are now in position to express the homomorphism $\phi_* : H_*(Z'; \mathbb{Z}) \to H_*(Z; \mathbb{Z})$ induced by the inclusion $Z' \subset Z$. By Lemma 1.3, one has a direct sum decomposition

$$H_*(Z'; \mathbb{Z}) = A_* \oplus B_* \,,$$

where

- A_* is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ long and $n \in J$.
- B_* is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ long and $n \notin J$.

Lemma 1.3 also gives a direct sum decomposition

$$H_*(Z;\mathbb{Z}) = A_* \oplus C_*\,,$$

where

- A_* is the free abelian group generated by $[W_J]$ with $J \subset \{1,\dots,n\}$ with $n \in J$ and J long.
- C_* is the free abelian group generated by $[W_J]$ with $J \subset \{1,\dots,n\}$ with $n \in J$ and J short.

Lemma 1.6. *(a) ϕ_* restricted to A_* coincides with the identity of A_*.*

(b) Suppose that ℓ is dominated. Then $\phi_*(B_*) \subset A_*$.

Proof. Point (a) is obvious. For (b), let $[W_J] \in B_{(n-|J|)(d-1)}$. By what has been seen, it suffices to show that $[W_J] \cdot [W_K]^\sharp = 0$ for all $[W_K] \in C_*$ with $|K| = |J|$. Suppose that there exists $[W_K] \in C_*$ with $|K| = |J|$ such that $[W_J]\cdot[W_K]^\sharp = \pm 1$. One has $[W_K]^\sharp = [W_L]$ where $L = \bar{K} \cup \{n\}$. By Lemma 1.4, this means that $J \cap (\bar{K} \cup \{n\}) = J - K = \{i\}$, with $i < n$. As $|K| = |J|$, this is equivalent to $K = (J - \{i\}) \cup \{n\}$. As $\ell_n \geq \ell_j$, this contradicts the fact that J is long and K is short. □

2 The Betti numbers of the chain space

Let $\ell = (\ell_1,\dots,\ell_n)$ be a dominated length vector. Let $a_k = a_k(\ell)$ be the number of short subsets J containing n with $|J| = k+1$. Alternatively, a_k is the number of sets $J \in \dot{\mathcal{S}}$ with $|J| = k$.

Theorem 2.1. *Let $\ell = (\ell_1,\dots,\ell_n)$ be a dominated length vector. Then, if $d \geq 3$, $H^*(\mathcal{C}_d^n(\ell);\mathbb{Z})$ is free abelian with rank*

$$\operatorname{rank} H^k(\mathcal{C}_d^n(\ell);\mathbb{Z}) = \begin{cases} a_s & \textit{if} \quad k = s(d-1), \quad s = 0,1,\dots,n-3, \\ a_{n-s-2} & \textit{if} \quad k = s(d-1)-1, \quad s = 0,\dots,n-2, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let N be a closed tubular neighbourhood of $\mathcal{C} = \mathcal{C}_d^n(\ell)$ in $Z = Z_d^n$. Let $Z' = Z - \mathcal{C}$. By Poincaré-Lefschetz duality and excision, one has the isomorphisms on integral homology

$$H^k(\mathcal{C}) \approx H^k(N) \approx H_{(n-1)(d-1)-k}(N,\partial N) \approx H_{(n-1)(d-1)-k}(Z,Z')$$

and

$$\begin{aligned} H^k(Z,\mathcal{C}) &\approx H^k(Z,N) \approx H^k(Z-\operatorname{int} N, \partial N) \\ &\approx H_{(n-1)(d-1)-k}(Z-\operatorname{int} N) \approx H_{(n-1)(d-1)-k}(Z')\,. \end{aligned}$$

The homology of Z and Z' are concentrated in degrees which are multiples of $(d-1)$. Hence, $H^k(\mathcal{C}) = 0$ if $k \not\equiv 0, -1 \mod (d-1)$. The possibly non-vanishing $H^k(\mathcal{C})$ sit in a diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^{s(d-1)-1}(\mathcal{C}) & \longrightarrow & H^{s(d-1)}(Z,\mathcal{C}) & \longrightarrow & H^{s(d-1)}(Z) & \longrightarrow & H^{s(d-1)}(\mathcal{C}) & \longrightarrow & 0 \\ & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \\ 0 & \longrightarrow & H_{r(d-1)+1}(Z,Z') & \longrightarrow & H_{r(d-1)}(Z') & \xrightarrow{\phi_{r(d-1)}} & H_{r(d-1)}(Z) & \longrightarrow & H_{r(d-1)}(Z,Z') & \longrightarrow & 0 \end{array}$$

with $r = n-1-s$. The horizontal sequences are exact. The (co)homology is with integral coefficients and the diagram commutes up to sign [1, Theorem I.2.2].

We deduce that $H_{r(d-1)}(Z,Z') \approx \operatorname{coker} \phi_{r(d-1)}$ which is isomorphic to $C_{r(d-1)}$ by Lemma 1.6. Therefore, $H^{s(d-1)}(\mathcal{C}^n_d(\ell))$ is free abelian with rank

$$\operatorname{rank} H^{s(d-1)}(\mathcal{C}^n_d(\ell)) = \operatorname{rank} C_{(n-1-s)(d-1)} = a_s\,.$$

On the other hand, $H_{r(d-1)+1}(Z,Z') \approx \ker \phi_{r(d-1)}$ which, by Lemma 1.6 is isomorphic (though not equal, in general) to $B_{r(d-1)}$. Therefore, $H^{s(d-1)-1}(\mathcal{C}^n_d(\ell))$ is free abelian with rank

$$\operatorname{rank} H^{s(d-1)-1}(\mathcal{C}^n_d(\ell)) = \operatorname{rank} B_{(n-1-s)(d-1)} = a_{n-s-2}\,.$$

□

Remark 2.2. Theorem 2.1 is wrong if ℓ is not dominated. For example, let $\ell = (1,1,1,\varepsilon)$ with $\varepsilon < 1$. Then, $\mathcal{C}^4_d(\ell)$ is diffeomorphic to the unit tangent bundle T^1S^{d-1} of S^{d-1}: a map $g\colon \mathcal{C}^4_d(\ell) \to T^1S^{d-1}$ is given by $g(\rho) = (\rho(1), \hat{\rho}(2))$, where the latter is obtained from $(\rho(1), \rho(2))$ by the Gram-Schmidt orthonormalization process. The map g is clearly a diffeomorphism for $\varepsilon = 0$ and the robot arm $F_{(1,1,1)}\colon \mathbb{S}^3_d \to \mathbb{R}^d$ of Section 1 has no critical value in the disk $\{|x| < 1\} \subset \mathbb{R}^d$. In particular, $\mathcal{C}^4_3(\ell)$ is diffeomorphic to $SO(3)$, and thus $H^2(\mathcal{C}^4_3(\ell);\mathbb{Z}) = \mathbb{Z}_2$. What goes wrong is Point (b) of Lemma 1.6: for instance $A_2 = 0$, $B_2 = H_2(Z,Z') \approx H^2(Z) = C_2 \approx \mathbb{Z}^3$ and, by the proof of Theorem 2.1, $\phi\colon H^2(Z') \to H^2(Z)$ must be injective with cokernel $\mathbb{Z}_2$. To obtain this fine result with our technique would require to control the signs in Lemma 1.4.

3 The manifold $V_d(\ell)$

Let $\ell \in \mathbb{R}^n_{>0}$ be a length vector. In [7, 8], a manifold $V_d(\ell)$ is introduced, whose boundary is $\mathcal{C} = \mathcal{C}^n_d(\ell)$, and Morse Theory on $V_d(\ell)$ provides some information on $\mathcal{C}$. In this section, we further study the manifold $V_d(\ell)$ in order to compute the ring $H^{(d-1)*}(\mathcal{C})$ when $d \geq 3$.

Presented as a submanifold of $Z = Z^n_d$, the manifold $V_d(\ell)$ is

$$V_d(\ell) = \{\rho \in Z \mid \sum_{i=1}^{n-1} \ell_i \rho(i) = te_1 \text{ with } t \geq \ell_n\}\,.$$

Observe that $V_d(\ell)$ is $O(d-1)$-invariant. Let $g : V_d(\ell) \to \mathbb{R}$ defined by $g(z) = -|\sum_{i=1}^{n-1} \ell_i z_i|$. The following proposition is proven in [7, Th. 3.2].

Proposition 3.1. *Suppose that the length vector $\ell \in \mathbb{R}^n_{>0}$ is generic. Then*

(i) $V_d(\ell)$ is a smooth $O(d-1)$-submanifold of Z, of dimension $(n-2)(d-1)$, with boundary $\mathcal{C}$.

(ii) $g\colon V_d(\ell) \to \mathbb{R}$ is an $O(d-1)$-equivariant Morse function, with critical points $\{\rho_J \mid J$ short and $n \in J\}$ (see (4) for the definition of ρ_J). The index of ρ_J is equal to $(d-1)(|J|-1)$. □

Corollary 3.2. *The cohomology group $H^*(V_d(\ell);\mathbb{Z})$ is free abelian and*

$$\operatorname{rank} H^k(V_d(\ell);\mathbb{Z}) = \begin{cases} a_s & \text{if } \; k = s(d-1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The number of critical point of g is equal to a_s. Corollary 3.2 is then obvious if $d \geq 3$. When $d = 2$, one uses [5, Theorem 4], the Morse function g being τ-invariant and its critical set being the the fixed point set $V_d(\ell)^\tau$. □

For each $J \subset \{1, \dots, n-1\}$, define the submanifold $\mathcal{R}_d(J)$ of $Z^n_d = Z$ by

$$\mathcal{R}_d(J) = \{\rho \in Z \mid \rho(i) = e_1 \text{ if } i \notin J\} \approx (S^{d-1})^J\,.$$

Consider the space

$$\mathcal{R}_d(\ell) = \bigcup_{J \in \dot{\mathcal{S}}} \mathcal{R}_d(J) \subset Z\,.$$

As $\dot{\mathcal{S}}$ is a simplicial complex, the family $\{[\mathcal{R}_d(J)] \mid J \in \dot{\mathcal{S}}\}$ is a free basis for $H_*(\mathcal{R}_d(\ell);\mathbb{Z})$ (homology classes of $\mathcal{R}_d(J)$ in lower degrees coincide in $H_*(\mathcal{R}_d(\ell))$ with $[\mathcal{R}_d(J')]$ for $J' \subset J$). Thus, $H_*(\mathcal{R}_d(\ell))$ is free abelian and

$$\operatorname{rank} H_k(\mathcal{R}_d(\ell);\mathbb{Z}) = \begin{cases} a_s & \text{if} \quad k = s(d-1) \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Lemma 3.3. *For $d \geq 2$, there exists a map $\mu: \mathcal{R}_d(\ell) \to V_d(\ell)$ such that $H^*\mu: H^*(V_d(\ell);\mathbb{Z}) \to H^*(\mathcal{R}_d(\ell);\mathbb{Z})$ is a ring isomorphism.*

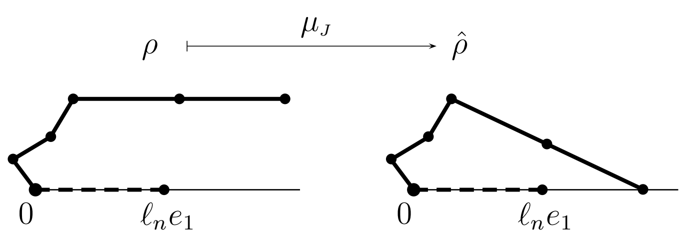
Proof. Let $J \in \dot{\mathcal{S}}$. Elementary Euclidean geometry shows that, for $\rho \in \mathcal{R}_d(J)$, there is a unique $\hat{\rho} \in V_d(\ell)$ satisfying the three conditions

(a) $\hat{\rho}(i) = \rho(i)$ if $i \in J$ and

(b) $|\hat{\rho}(\bar{J})| = 1$, where $\bar{J} = \{1, \dots, n-1\} - J$.

(c) $\langle \hat{\rho}(i), e_1 \rangle > 0$ if $i \in \bar{J}$.

This defines an embedding $\mu_J: \mathcal{R}_d(J) \to V_d(\ell)$ by $\mu_J(\rho) = \hat{\rho}$. An example is drawn below with $n = 6$ and $J = \{1, 2, 3\}$ (the last segments $\ell_n \rho(n) = -\ell_n e_1$ of the configurations are not drawn).



We shall construct the map $\mu: \mathcal{R}_d(\ell) \to V_d(\ell)$ so that its restriction to $\mathcal{R}_d(J)$ is homotopic to μ_J for each $J \in \dot{\mathcal{S}}$. Unfortunately, when $J \subset J'$, the restriction of $\mu_{J'}$ to $\mathcal{R}_d(J)$ is not equal to μ_J so the construction of μ requires some work.

For $J \in \dot{\mathcal{S}}$, consider the space of embeddings

$$\mathfrak{N}_J = \{\alpha: \mathcal{R}_d(J) \to V_d(\ell) \mid \alpha(\rho) \text{ satisfies (a) and (c)}\}$$

We claim that $\mathfrak{N}_J$ retracts by deformation onto its one-point subspace $\{\mu_J\}$. Indeed, let $\alpha \in \mathfrak{N}_J$ and let $\rho \in \mathcal{R}_d(\ell)$. For $J \in \dot{\mathcal{S}}$, consider the space

$$A_\rho = \{\zeta \colon \bar{J} \to S^{d-1} \mid \langle \zeta(i), e_1\rangle > 0 \text{ and } \sum_{i\in J} \rho(i) + \sum_{i \in \bar{J}} \zeta(i) = \lambda e_1 \text{ with } \lambda > 0\}\,.$$

Obviously, $\alpha(\rho)_{|\bar{J}} \in A_\rho$. The space A_ρ is a submanifold of $(S^{d-1})^{|\bar{J}|}$ which can be endowed with the induced Riemannian metric. The parameter λ provides a function $\lambda\colon A_\rho \to \mathbb{R}$. As usual, this is a Morse function with critical points the lined configurations $\zeta(i) = \pm\zeta(j)$. But, as $\langle\zeta(i), e_1\rangle > 0$, the only critical point is a maximum, the restriction of $\mu_J(\rho)$ to $\bar{J}$. Following the gradient line at constant speed thus produces a deformation retraction of A_ρ onto $\mu_J(\rho)_{|\bar{J}}$. The manifold A_ρ and its gradient vector field depending smoothly on ρ, this provides the required deformation retraction of $\mathfrak{N}_J$ onto $\{\mu_j\}$.

Let $\mathcal{BS}_n$ be the first barycentric subdivision of $\dot{\mathcal{S}}$. Recall that the vertices of $\mathcal{BS}_n$ are the barycenters $\hat{J} \in |\dot{\mathcal{S}}|$ of the simplexes J of $\dot{\mathcal{S}}$, where $|\cdot|$ denotes the geometric realization. A family $\{\hat{J}_0, \dots, \hat{J}_k\}$ of distinct barycenters forms a k-simplex $\sigma \in \mathcal{BS}_n$ if $J_0 \subset J_1 \subset \cdots \subset J_k$. Set $\min\sigma = J_0$. For $J \in \dot{\mathcal{S}}$, we also see $\hat{J}$ as a point of $|\mathcal{BS}_n| = |\dot{\mathcal{S}}|$.

Let us consider the quotient space:

$$\hat{\mathcal{R}}_d(\ell) = \coprod_{\sigma \in \mathcal{BS}_n} |\sigma| \times \mathcal{R}_d(\min\sigma) \Big/ \sim\,, \tag{8}$$

where $(t, \rho) \sim (t', \rho')$ if $\sigma < \sigma'$, $t = t' \in |\sigma| \subset |\sigma'|$ and $\rho \mapsto \rho'$ under the inclusion $\mathcal{R}_d(\min\sigma) \hookrightarrow \mathcal{R}_d(\min\sigma')$. The projections onto the first factors in (8) provide a map $q\colon \hat{\mathcal{R}}_d \to |\mathcal{BS}_n|$ such that $q^{-1}(\hat{J}) = \{\hat{J}\} \times \mathcal{R}_d(J) \approx \mathcal{R}_d(J)$. Over a 1-simplex $e = \{\{J\}, \{J, J'\}\}$ of $\mathcal{BS}_n$, one has $q^{-1}(\{J\}) \approx \mathcal{R}_d(J)$, $q^{-1}(\{J'\}) \approx \mathcal{R}_d(J')$ and $q^{-1}(|e|)$ is the mapping cylinder of the inclusion $\mathcal{R}_d(J) \hookrightarrow \mathcal{R}_d(J')$.

We now define a map $\hat{\mu}\colon \hat{\mathcal{R}}_d \to V_d(\ell)$ by giving its restriction

$$\hat{\mu}^k \colon q^{-1}(|\mathcal{BS}_d(\ell)^k| \to V_d(\ell)$$

over the k-skeleton of $\mathcal{BS}_n$. We proceed by induction on k. The restriction of $\hat{\mu}$ to $q^{-1}(\hat{J}) = \mathcal{R}_d(J)$ is equal to $\mu_J \in \mathfrak{N}_J$. This defines $\hat{\mu}^0$. For an edge $e = \{\{J\}, \{J, J'\}\}$, we use that $\mathfrak{N}_J$ is contractible, as seen above. The restriction of $\mu_{J'}$ to $\mathcal{R}_d(J)$ is thus homotopic to μ_J and we can use a homotopy to extend $\hat{\mu}^0$ over $|e|$. Thus $\hat{\mu}^1$ is defined. Suppose that $\hat{\mu}^{k-1}$ is defined. Let

$\sigma = \{\hat{J}_0, \dots, \hat{J}_k\}$ be a a k-simplex of $\mathcal{BS}_d(\ell)$ with $\min\sigma = J_0$ and with boundary $\mathrm{Bd}\,\sigma$. As $\mathfrak{N}_{J_0}$ is contractible, the restriction of $\hat{\mu}^{k-1}$ to $q^{-1}(|\mathrm{Bd}\,\sigma|)$ extends to $q^{-1}(|\sigma|)$. This process permits us to define $\hat{\mu}^k$.

Now the projections to the second factors in (8) give rise to a surjective map $p\colon \hat{\mathcal{R}}_d(\ell) \to \mathcal{R}_d(\ell)$. Let $x \in \mathcal{R}_d(\ell)$. Let $J \in \dot{\mathcal{S}}$ minimal such that $x \in \mathcal{R}_d(J)$. Then

$$p^{-1}(\{x\}) = |\mathrm{Star}(\hat{J}, \mathcal{BS}_n)| \times \{x\}$$

is a contractible polyhedron. The preimages of points of p are then all contractible and locally contractible, which implies that p is a homotopy equivalence [12]. Using a homotopy inverse for p and the map $\hat{\mu}$, we get a map $\mu\colon \mathcal{R}_d(\ell) \to V_d(\ell)$.

For $J \in \dot{\mathcal{S}}$, let us compose μ_J with the inclusion $\beta\colon V_d(\ell) \hookrightarrow Z$. When $\rho \in \mathcal{R}_d(J)$, the common value $\hat{\rho}(i)$ for $i \notin J$ is not equal to $-(e_1, e_1, \dots, e_1)$. Using arcs of geodesics in S^{d-1} enables us to construct a homotopy from $\beta\circ\mu_J$ to the inclusion of $\mathcal{R}_d(J)$ into Z. This implies that $H_*\mu\colon H_*(\mathcal{R}_d(\ell);\mathbb{Z}) \to H_*(V_d(\ell);\mathbb{Z})$ is injective. By Corollary 3.2 and (7), $H_*\mu$ is an isomorphism. Hence, $H^*\mu\colon H^*(V_d(\ell);\mathbb{Z}) \to H^*(\mathcal{R}_d(\ell);\mathbb{Z})$ is a ring isomorphism. □

Remark 3.4. When $d \geq 3$, Lemma 3.3 implies that $\mu\colon \mathcal{R}_d(\ell) \to V_d(\ell)$ is a homotopy equivalence, since the spaces under consideration are simply connected. We do not know if μ is also a homotopy equivalence when $d = 2$.

4 Proof of Theorm B

Theorem B is a direct consequence of Propositions 4.1 and 4.3 below.

Proposition 4.1. *Let $\ell \in \mathbb{R}^n_{>0}$ be a generic length vector which is dominated. Then the inclusion $\mathcal{C}^n_d(\ell) \subset V_d(\ell)$ induces an injective ring homomorphism*

$$H^*(\mathcal{R}_d(\ell);\mathbb{Z}) \approx H^*(V_d(\ell);\mathbb{Z}) \hookrightarrow H^*(\mathcal{C}^n_d(\ell);\mathbb{Z})\,. \tag{9}$$

When $d \geq 3$ its image is equal to the subring $H^{(d-1)*}(\mathcal{C}^n_d(\ell);\mathbb{Z})$.

Proof. By Theorem 2.1 and its proof, the homomorphism $H^{s(d-1)}(Z^n_d;\mathbb{Z}) \to H^{s(d-1)}(\mathcal{C}^n_d(\ell);\mathbb{Z})$ induced by the inclusion is surjective and $\mathrm{rank}\, H^{s(d-1)}(\mathcal{C}^n_d(\ell);\mathbb{Z}) = a_s$ (recall that $Z^n_d = Z$). As the inclusion $\mathcal{C}^n_d(\ell) \subset Z^n_d$ factors through $V_d(\ell)$ the homomorphism $H^{s(d-1)}(V_d(\ell);\mathbb{Z}) \to H^{s(d-1)}(\mathcal{C}^n_d(\ell);\mathbb{Z})$ induced by the inclusion is also surjective. As $\mathrm{rank}\, H^{s(d-1)}(V_d(\ell);\mathbb{Z}) = a_s$ by Corollary 3.2, this proves the proposition. □

Remark 4.2. Proposition 4.1 is wrong if ℓ is not dominated. For example, let $\ell = (1,1,1,\varepsilon)$ with $\varepsilon < 1$. Then $a_1 = 3$, so $H^{d-1}(V_d(\ell);\mathbb{Z}) \approx \mathbb{Z}^3$. But, for $d = 3$, we saw in Remark 2.2 that $H^2(\mathcal{C}_3^4(\ell);\mathbb{Z}) = \mathbb{Z}_2$.

As in the introduction, consider the polynomial ring $\mathbb{Z}_2[X_1,\dots,X_{n-1}]$ with formal variables $X_1,\dots,X_{n-1}$. If $J \subset \{1,\dots,n-1\}$, we denote by $X_J \in \mathbb{Z}_2[X_1,\dots,X_{n-1}]$ the monomial $\prod_{j\in J} X_j$. Let $\mathcal{I}(\tilde{\mathcal{S}}(\ell))$ be the ideal of $\mathbb{Z}_2[X_1,\dots,X_{n-1}]$ generated by the squares X_i^2 of the variables and the monomials X_J for $J \notin \tilde{\mathcal{S}}(\ell)$ (non-simplex monomials).

Proposition 4.3. *The ring $H^*(\mathcal{R}_d(\ell);\mathbb{Z}_2)$ is isomorphic to the quotient ring $\mathbb{Z}_2[X_1,\dots,X_{n-1}]/\mathcal{I}(\tilde{\mathcal{S}}(\ell))$ (The degree of X_i being $d-1$).*

Proof. The coefficients of the (co)homology groups are $\mathbb{Z}_2$ and are omitted in the notation. Consider the inclusion $\beta\colon V_d(\ell) \hookrightarrow Z = Z_d^n$. The map $\rho \mapsto (\rho(1),\dots,\rho(n-1))$ is a diffeomorphism from Z to $(S^{d-1})^{n-1}$. Using this identification, the homology $H_*(Z)$ is the $\mathbb{Z}_2$-vector space with basis the classes $[\mathcal{R}_d(I))]$ for $I \subset \{1,\dots,n-1\}$. (To compare with the basis of Lemma 1.3, the submanifolds $R_d(J)$ and $W_{\bar{J}}$ are isotopic, where $\bar{J}$ is the complement of J in $\{1,\dots,n\}$.) The homology $H_*(\mathcal{R}_d(\ell))$ has basis $[\mathcal{R}_d(J)]$ for $J \in \tilde{\mathcal{S}}(\ell)$. The homomorphism $H_*\beta\colon H_*(\mathcal{R}_d(\ell)) \to H_*(Z)$ is induced by the inclusion of the above bases. Hence, $H_j\beta\colon H_j(\mathcal{R}_d(\ell)) \to H_j(Z)$ is injective and $\operatorname{coker} H_j$ is freely generated by the classes $[\mathcal{R}_d(J)]$ for $|J| = j$ and $J \notin \tilde{\mathcal{S}}(\ell)$.

In particular, the classes $[\mathcal{R}_d(\{i\})]$, for $i = 1,\dots,n-1$, form a basis of $H_{d-1}(Z)$. Let $\{\xi_1,\dots,\xi_{n-1}\} \in H^{d-1}(Z) = \hom(H_{d-1}(Z),\mathbb{Z}_2)$ be the Kronecker dual basis. By the Künneth formula, the correspondence $X_i \mapsto \xi_i$ extends to a ring isomorphism $\mathbb{Z}_2[X_1,\dots,X_{n-1}] \xrightarrow{\approx} H^*(Z)$. The the family of monomials $\{X_J \mid J \subset \{1,\dots,n-1\}\}$ is sent to the the Kronecker dual basis to $\{[\mathcal{R}_d(J)] \mid J \subset \{1,\dots,n-1\}\}$. The properties of $H_*\beta$ mentioned above then imply that the composed ring homomorphism

$$\mathbb{Z}_2[X_1,\dots,X_{n-1}] \xrightarrow{\approx} H^*(Z) \xrightarrow{H^*\beta} H^*(\mathcal{R}_d(\ell))$$

is surjective with kernel $\mathcal{I}(\tilde{\mathcal{S}}(\ell))$. □

The proof of Theorem B is thus complete which, as seen in the introduction, implies Theorem A.

5 Comments

5.1. The authors are trying to unify the notations used for the various posets of short subsets. Our notation $\tilde{\mathcal{S}} \subset \dot{\mathcal{S}} \subset \mathcal{S}$ are identical to that of [11]. In [9], $\dot{\mathcal{S}}$ is denoted by $\mathcal{S}_n$ but, in the more recent papers [10, 8], $\mathcal{S}_n = \{J \in \mathcal{S} \mid n \in J\}$. This is not used here but could have been naturally in e.g. Theorem 2.1.

5.2. When $d = 2$, Assertion (9) still holds true but not the last assertion of Proposition 4.1. The image $\mathcal{J}_2^n(\ell)$ of the homomorphism $H^*(V_2(\ell); \mathbb{Z}) \to H^*(\mathcal{C}_2^n(\ell); \mathbb{Z})$ induced by the inclusion is just some subring of $H^*_{(1)}(\mathcal{C}_2^n(\ell); \mathbb{Z})$, where the latter denotes the subring of $H^*(\mathcal{C}_2^n(\ell); \mathbb{Z})$ generated by the elements of degree 1. For length vectors such that $\mathcal{J}_2^n(\ell) = H^*_{(1)}(\mathcal{C}_2^n(\ell); \mathbb{Z})$, our proof of Theorem B (and then of Theorem A) holds. Such length vectors are called normal in [4].

5.3. The ring structure of $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z}_2)$ is necessary to differentiate the chain spaces up to diffeomorphism: the Betti numbers are not enough. The first example occurs for $n = 6$ with $\ell = (1, 1, 1, 2, 3, 3)$ and $\ell' = (\varepsilon, 1, 1, 1, 2, 2)$, where $0 < \varepsilon < 1$. (The chamber of ℓ is $\langle 632, 64 \rangle$ and that of ℓ' is $\langle 641 \rangle$, see [8, Table C].) Then, $\tilde{\mathcal{S}}(\ell)$ and $\tilde{\mathcal{S}}(\ell')$ are both graphs with 4 vertices and 3 edges. Therefore, $a_s(\ell) = a_s(\ell')$ for all s which, by Theorem 2.1, implies that $\mathcal{C}_d^6(\ell)$ and $\mathcal{C}_d^6(\ell')$ have the same Betti numbers. However, $\tilde{\mathcal{S}}(\ell)$ and $\tilde{\mathcal{S}}(\ell')$ are not poset isomorphic: the former is not connected while the latter is.

5.4. It would be interesting to know if, in Theorem A, the ring $\mathbb{Z}_2$ could be replaced by any other coefficient ring. In the corresponding result for spatial polygon spaces $\mathcal{N}_3^n(\ell)$, which are distinguished by their $\mathbb{Z}_2$-cohomology rings if $n > 4$ [4, Theorem 3], the ring $\mathbb{Z}_2$ cannot be replaced by $\mathbb{R}$. Indeed, $\mathcal{N}_3^5(\varepsilon, 1, 1, 1, 2) \approx \mathbb{C}P^2 \sharp \bar{\mathbb{C}}P^2$ while $\mathcal{N}_3^5(\varepsilon, \varepsilon, 1, 1, 1) \approx S^2 \times S^2$ (ε small; see [8, Table B]). These two manifolds have non-isomorphic $\mathbb{Z}_2$-cohomology rings but isomorphic real cohomology rings. One can of course replace $\mathbb{Z}_2$ by $\mathbb{Z}$ in Theorem A since, by Theorem 2.1, $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z})$ determines $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z}_2)$ when ℓ is dominated.

5.5. We do not know if Theorem A is true for generic length vectors which are not dominated. The techniques developped in [3] might useful to study this more general case.

5.6. Let K be a flag simplicial complex (i.e. if K contains a graph L isomorphic to the 1-skeleton of a r-simplex, then L is contained in a r-simplex of K). Then the complex $\mathcal{R}_1(K)$ is the Salvetti complex of the right-angled Coxeter group determined by the 1-skeleton of K, see [2].

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