ON THE DIRECT PRODUCT CONJECTURE FOR SIGMA INVARIANTS

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Abstract. We show that the direct product conjecture for $\Sigma^n(G;\mathbb{Z})$, where G is the direct product of two groups of type FP_n holds for $n = 3$ and give counterexamples for $n \geq 4$. Previously counterexamples were only known for a related conjecture involving the homotopical Σ-invariants, where the conjecture already fails for $n \geq 3$.

1. Introduction

A group G is said to be of type FP_n, if there is a resolution

$$\ldots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{1}$$

of free $\mathbb{Z}G$-modules such that F_i is finitely generated for $i \leq n$. Here $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}G$-module. Bieri and Renz [5], building on work of [4], have introduced certain geometric invariants of G which contain information on such finiteness properties of subgroups of G which occur as kernels of real-valued homomorphisms.

Given a group of type FP_n with $n \geq 1$, these invariants are defined as follows. We first define

$$S(G) \quad = \quad (\mathrm{Hom}(G,\mathbb{R}) - \{0\})/\mathbb{R}_+,$$

that is, we identify nonzero homomorphisms, if one is a positive multiple of the other. This is a sphere of dimension $\mathrm{rank}(G/[G,G]) - 1$. If $\chi : G \to \mathbb{R}$ is a nonzero homomorphism, we still write $\chi \in S(G)$.

Given such χ, we let $G_\chi = \{g \in G \,|\, \chi(g) \geq 0\}$. If there is a resolution (1) of free $\mathbb{Z}G_\chi$ modules with F_i finitely generated for $i \leq k$, we say G_χ *is of type* FP_k. We now set

$$\Sigma^k(G;\mathbb{Z}) \quad = \quad \{\chi \in S(G) \,|\, G_\chi \text{ is of type } FP_k\}.$$

If $G = G_1 \times G_2$ and $\chi : G \to \mathbb{R}$ a homomorphism, we obtain homomorphisms $\chi_i : G_i \to \mathbb{R}$ for $i = 1, 2$ by restriction. For a direct product of groups the following conjecture has been formulated, see Bieri [2] or Meinert [9]. Notice that the zero homomorphism is not contained in any $\Sigma^n(G;\mathbb{Z})$.

Conjecture 1. *Let G_1 and G_2 be groups of type FP_n. Then $\chi \notin \Sigma^n(G_1 \times G_2;\mathbb{Z})$ if and only if $\chi_1 \notin \Sigma^p(G_1;\mathbb{Z})$ and $\chi_2 \notin \Sigma^q(G_2;\mathbb{Z})$ for some p and q with $p + q = n$.*

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For $n = 1$ this is proven in Bieri, Neumann and Strebel [4] and the case $n = 2$ has been established by Gehrke [7]. Gehrke [7], based on work of Meinert, also showed that $\chi \notin \Sigma^n(G_1 \times G_2; \mathbb{Z})$ implies $\chi_1 \notin \Sigma^p(G_1; \mathbb{Z})$ and $\chi_2 \notin \Sigma^q(G_2; \mathbb{Z})$ for some p and q with $p + q = n$ for all n.

If for a given $\chi_1 : G_1 \to \mathbb{R}$ and $\chi_2 : G_2 \to \mathbb{R}$ Conjecture 1 holds, we say that χ_1 and χ_2 satisfy the product formula.

The conjecture is often expressed in terms of subsets of spheres as, for example, in [2, 8, 9]. It then takes the following form. Here $\Sigma^n(G; \mathbb{Z})^c$ denotes the complement of $\Sigma^n(G; \mathbb{Z})$ in $S(G)$.

Conjecture 2. *Let G_1 and G_2 be groups of type FP_n. Then*

$$\Sigma^n(G_1 \times G_2; \mathbb{Z})^c = \bigcup_{p+q=n} \Sigma^p(G_1; \mathbb{Z})^c * \Sigma^q(G_2; \mathbb{Z})^c$$

where $*$ stands for the join in $S(G_1 \times G_2) = S(G_1) * S(G_2)$.

A similar conjecture involving homotopical sigma invariants $\Sigma^n(G)$, see [5, Rmk.6.5] for the definition, has been shown to be false for $n \geq 3$ by Meier, Meinert and van Wyk [8, §6]. Again this conjecture was known to hold for $n \leq 2$.

The example of [8] exploits the subtle differences between groups of type FP_m and groups of type F_m, that is, groups for which there exists a $K(G, 1)$ with finite m-skeleton, which do not apply in the homological version. The following Theorem now settles Conjecture 1.

Theorem 1.1. *Conjecture 1 is true for $n \leq 3$, but there exist counterexamples for $n \geq 4$.*

Notice that the conjecture holds for $n = 3$ despite the counterexamples to the homotopical conjecture for $n = 3$. The counterexamples for $n \geq 4$ are, as the examples of [8], based on the work of Bestvina and Brady [1] on right-angled Artin groups. They are described in Example 3.7.

The proof of Theorem 1.1 is based on a description of the Sigma-invariants in terms of vanishing Novikov homology groups. This is explained in Section 2. Using a Künneth formula for these homology groups we obtain vanishing and non-vanishing results for the Novikov homology of the direct product $G_1 \times G_2$. In particular, this gives an alternative approach to the results of Gehrke and Meinert mentioned above. Finally, in the last section we consider the case where one of the groups is a right-angled Artin group and describe conditions on this group so that the product formula holds.

In [2, Thm.3.2], Bieri and Geoghegan announce a result which gives an alternative proof of the case $n = 3$, though the proof has yet to appear. It is worth pointing out that in [2] they also announce a proof of Conjecture 1 in case $\mathbb{Z}$ is replaced by a field $\mathbf{k}$. The author would like to thank Robert Bieri and Ross Geoghegan for valuable comments and information about their unpublished work.

2. Homological criteria for Sigma invariants

In this section we want to describe a criterion for the sigma invariants based on Novikov homology. We start with the definition of the Novikov ring.

Let G be a group and $\chi : G \to \mathbb{R}$ a homomorphism. We denote by $\mathbb{Z}^G$ the abelian group of all functions $\lambda : G \to \mathbb{Z}$. For $\lambda \in \mathbb{Z}^G$ denote $\operatorname{supp}\lambda = \{g \in G \mid \lambda(g) \neq 0\}$.

Definition 2.1. The *Novikov ring* $\widehat{\mathbb{Z}G}_\chi$ is defined as

$$\widehat{\mathbb{Z}G}_\chi \quad = \quad \{\lambda \in \mathbb{Z}^G \mid \forall\, r \in \mathbb{R} \quad \operatorname{supp}\lambda \cap \xi^{-1}((-\infty, r]) \text{ is finite}\}$$

The multiplication is given by the extension of the multiplication of the group ring.

If G_1 and G_2 are groups, and $G = G_1 \times G_2$, it is clear that $\mathbb{Z}G = \mathbb{Z}G_1 \otimes \mathbb{Z}G_2$. Now if $\chi_i : G_i \to \mathbb{R}$ are homomorphisms for $i = 1, 2$, we can form $\chi : G \to \mathbb{R}$ by $\chi(g_1, g_2) = \chi_1(g_1) + \chi_2(g_2)$. There is an inclusion $\widehat{\mathbb{Z}G_1}_{\chi_1} \otimes \widehat{\mathbb{Z}G_2}_{\chi_2} \to \widehat{\mathbb{Z}G}_\chi$, but this is in general not an isomorphism. In particular if both homomorphisms are nonzero, it is not an isomorphism. This will lead to some extra complication in Section 4.

The following lemma is well-known, a proof can be found in Bieri [3, Thm.A.1].

Lemma 2.2. *Let G be a group of type FP_n and $k \leq n$. Then the following are equivalent.*

1. $\chi \in \Sigma^k(G; \mathbb{Z})$.
2. $\operatorname{Tor}_i^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) = 0$ *for* $i \leq k$. □

If X is a $K(G, 1)$, we also write $H_i(X; \widehat{\mathbb{Z}G}_\chi) = \operatorname{Tor}_i^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z})$ for $i \in \mathbb{Z}$.

3. The Künneth Formula for Novikov homology

As $\widehat{\mathbb{Z}G}_\chi \neq \widehat{\mathbb{Z}G_1}_{\chi_1} \otimes \widehat{\mathbb{Z}G_2}_{\chi_2}$, we need a lemma on change of coefficients.

Lemma 3.1. *Let R be a ring and C_* a left chain complex over R of flat modules with $C_i = 0$ for $i < 0$ and $\rho : R \to S$ a ring homomorphism. If $H_i(C) = 0$ for $i < k$, then $H_i(S \otimes_R C) = 0$ for $i < k$ and $H_k(S \otimes_R C) = S \otimes_R H_k(C)$.*

Proof. This follows directly from the Universal Coefficient Spectral Sequence. □

Lemma 3.2. *Let C_* be a free left chain complex over R_1 with $C_i = 0$ for $i < 0$ and D_* a free left chain complex over R_2 with $D_i = 0$ for $i < 0$. Assume also that R_1 or R_2 is torsionfree as an abelian group. If $H_i(C) = 0$ for $i < k$ and $H_i(D) = 0$ for $i < l$, then $H_i(C \otimes_{\mathbb{Z}} D) = 0$ for $i < k + l$ and*

$$H_{k+l}(C \otimes_{\mathbb{Z}} D) \quad = \quad H_k(C) \otimes_{\mathbb{Z}} H_l(D)$$

as $R_1 \otimes_{\mathbb{Z}} R_2$-modules.

Proof. As one of R_1 or R_2 is a torsionfree abelian group, the result in terms of abelian groups follows from the ordinary Künneth theorem. Furthermore, the resulting isomorphism of abelian groups

$$H_k(C) \otimes_{\mathbb{Z}} H_l(D) \longrightarrow H_{k+l}(C \otimes_{\mathbb{Z}} D)$$

is easily seen to respect the $R_1 \otimes_{\mathbb{Z}} R_2$-module structure. This gives the result. □

Corollary 3.3. *Let G_1 be a group of type FP_k and G_2 be a group of type FP_l with k, l positive integers. Let $\chi : G_1 \times G_2 \to \mathbb{R}$ be a homomorphism and χ_i the* *restriction to G_i for $i = 1, 2$. Assume that $\chi_1 \in \Sigma^{k-1}(G_1;\mathbb{Z})$ and $\chi_2 \in \Sigma^{l-1}(G_2;\mathbb{Z})$. Then*

$$\begin{aligned} \mathrm{Tor}_i^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) &= 0 \qquad \text{for } i \leq k+l-1 \\ \mathrm{Tor}_{k+l}^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) &\cong \widehat{\mathbb{Z}G}_\chi \otimes_R (\mathrm{Tor}_{k+l}^{\mathbb{Z}G_1}(\widehat{\mathbb{Z}G_1}_{\chi_1}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Tor}_{k+l}^{\mathbb{Z}G_2}(\widehat{\mathbb{Z}G_2}_{\chi_2}, \mathbb{Z})) \end{aligned}$$

where $G = G_1 \times G_2$ and $R = \widehat{\mathbb{Z}G_1}_{\chi_1} \otimes \widehat{\mathbb{Z}G_2}_{\chi_2}$.

Proof. If $C \to \mathbb{Z}$ is a resolution of $\mathbb{Z}$ of free $\mathbb{Z}G_1$-modules and $D \to \mathbb{Z}$ a resolution of free $\mathbb{Z}G_2$-modules, the double complex $C \otimes_{\mathbb{Z}} D$ gives a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The result then follows from Lemma 2.2, Lemma 3.2 and Lemma 3.1 as all of the involved rings are torsionfree as abelian groups. □

Let us also state a version of Corollary 3.3 in the case that G_1 and G_2 are of type F_n.

Corollary 3.4. *Let X be a $K(G_1, 1)$ and Y a $K(G_2, 1)$, both with finite n-skeleton, and let k, l be positive integers with $k + l \leq n$. Let $\chi : G_1 \times G_2 \to \mathbb{R}$ be a homomorphism and χ_i the restriction to G_i for $i = 1, 2$. Assume that $\chi_1 \in \Sigma^{k-1}(G_1;\mathbb{Z})$ and $\chi_2 \in \Sigma^{l-1}(G_2;\mathbb{Z})$. Then $\chi \in \Sigma^{k+l-1}(G;\mathbb{Z})$ and*

$$H_{k+l}(X \times Y; \widehat{\mathbb{Z}G}_\chi) \cong \widehat{\mathbb{Z}G}_\chi \otimes_R (H_k(X; \widehat{\mathbb{Z}G_1}_{\chi_1}) \otimes_{\mathbb{Z}} H_l(Y; \widehat{\mathbb{Z}G_2}_{\chi_2}))$$

where $G = G_1 \times G_2$ and $R = \widehat{\mathbb{Z}G_1}_{\chi_1} \otimes \widehat{\mathbb{Z}G_2}_{\chi_2}$.

Proof. The Eilenberg-Zilber chain map $C_*(\tilde{X}) \otimes_{\mathbb{Z}} C_*(\tilde{Y}) \to C_*(\tilde{X} \times \tilde{Y})$, where $\tilde{X}$ and $\tilde{Y}$ are the universal covering spaces of X and Y, is a chain homotopy equivalence of $\mathbb{Z}G$ chain complexes, so

$$H_*(X \times Y; \widehat{\mathbb{Z}G_1}_{\chi_1} \otimes \widehat{\mathbb{Z}G_2}_{\chi_2}) \cong H_*(C(X; \widehat{\mathbb{Z}G_1}_{\chi_1}) \otimes C(Y; \widehat{\mathbb{Z}G_2}_{\chi_2})).$$

Therefore the result follows from Lemma 3.2 and Lemma 3.1 as in Corollary 3.3. □

Corollary 3.3 gives an alternative proof of one implication of Conjecture 1, see Gehrke [7] for a different proof. It also indicates why the conjecture is not true in general: a tensor product of two non-trivial abelian groups can be trivial.

Corollary 3.5. *Let G_1 and G_2 be groups of type FP_n. Assume $\chi \notin \Sigma^n(G_1 \times G_2;\mathbb{Z})$. Then $\chi_1 \notin \Sigma^p(G_1;\mathbb{Z})$ and $\chi_2 \notin \Sigma^q(G_2;\mathbb{Z})$ for some p and q with $p + q = n$.*

Proof. Choose p with $\chi_1 \in \Sigma^{p-1}(G_1;\mathbb{Z}) - \Sigma^p(G_1;\mathbb{Z})$ and q with $\chi_2 \in \Sigma^{q-1}(G_2;\mathbb{Z}) - \Sigma^q(G_2;\mathbb{Z})$. Then $\chi \in \Sigma^{p+q-1}(G_1 \times G_2;\mathbb{Z})$ by Corollary 3.3, so $p+q \leq n$. If $p+q < n$ replace p by $n - q > p$. □

Let X be a finite CW-complex and $G = \pi_1(X)$. Given a nonzero homomorphism $\chi : G \to \mathbb{R}$ we can find a map $f : \tilde{X} \to \mathbb{R}$, where $\tilde{X}$ is the universal cover of X, satisfying $f(gx) = \chi(g) + f(x)$ for all $x \in \tilde{X}$ and $g \in G$. In this situation, we denote $N = f^{-1}([0, \infty))$.

The following Lemma is due to Sikorav [10], it follows from the fact that the Novikov ring can be interpreted as an inverse limit, compare also the proof of Lemma 4.2 below.

Lemma 3.6. *Let X be a finite CW complex, $\chi : \pi_1(X) \to \mathbb{R}$ a nonzero homomorphism, $\rho : \tilde{X} \to X$ the universal covering space and $G = \pi_1(X)$. Let $g \in G$ satisfy $\xi(g) > 0$ and let $q \in \mathbb{Z}$. Then there is a natural short exact sequence*

$$0 \longrightarrow \varprojlim{}^1 H_{q+1}(\tilde{X}, g^i N; \mathbb{Z}) \longrightarrow H_q(X; \widehat{\mathbb{Z}G}_\chi) \longrightarrow \varprojlim H_q(\tilde{X}, g^i N; \mathbb{Z}) \longrightarrow 0 \tag{2}$$

where the limits are taken over positive integers i and the maps are induced by inclusions. □

Given a simplicial complex L which is a flag complex, denote L^0 as the set of vertices and L^1 as the set of edges. Bestvina and Brady [1] define the right-angled Artin group corresponding to L by

$$G_L \quad = \quad \langle v_i \in L^0 \,|\, [v_i, v_j] \text{ for } (v_i, v_j) \in L^1 \rangle.$$

Example 3.7. Let p and q be positive integers with $(p, q) = 1$ and $n, m \geq 2$. Let $L_1 = M(\mathbb{Z}/p, n-1)$ and $L_2 = M(\mathbb{Z}/q, m-1)$ be Moore spaces, that is, L_1 is obtained from the $n-1$-sphere by attaching an n-cell with a map of degree p and similar for L_2. Subdivide L_1 and L_2 so that they are flag complexes. For $i = 1, 2$ let $G_i = G_{L_i}$ be the associated right angled Artin group and $\chi_i : G_i \to \mathbb{Z}$ the homomorphism sending every generator to 1. There exists a finite $K(G_i, 1)$ which we denote by Q_i for $i = 1, 2$, see, for example, [1]. By Bestvina and Brady [1, Thm.7.1] together with Lemma 3.6 we get

$$H_i(Q_1; \widehat{\mathbb{Z}G_1}_{\chi_1}) \quad \cong \quad \begin{cases} 0 & \text{for} \quad i \neq n \\ \varprojlim \bigoplus \mathbb{Z}/p & \text{for} \quad i = n \end{cases}$$

and

$$H_i(Q_2; \widehat{\mathbb{Z}G_2}_{\chi_2}) \quad \cong \quad \begin{cases} 0 & \text{for} \quad i \neq m \\ \varprojlim \bigoplus \mathbb{Z}/q & \text{for} \quad i = m \end{cases}$$

where both inverse systems are given by projections onto direct summands. In particular, we get that $H_n(Q_1; \widehat{\mathbb{Z}G_1}_{\chi_1}) \neq 0$ and is p-torsion as an abelian group, while $H_m(Q_2; \widehat{\mathbb{Z}G_2}_{\chi_2}) \neq 0$ and is q-torsion as an abelian group. As $(p, q) = 1$, we get $H_{n+m}(Q_1 \times Q_2; \widehat{\mathbb{Z}G_1 \times G_2}_{\chi_1+\chi_2}) = 0$ from Corollary 3.4. Therefore we have a counterexample to Conjecture 1 for $n \geq 4$.

Remark 3.8. Notice that $G = G_1 \times G_2$ in Example 3.7 is the right-angled Artin group corresponding to the join $L = L_1 * L_2$. It is easy to see that L is contractible, so the counterexample can be derived directly from the work of Bestvina and Brady. Nevertheless the Künneth formula will become useful in the next section.

4. Proof of the Conjecture for $n = 3$

We were able to find a counterexample by producing non-zero Novikov homology groups whose tensor product was zero. In this section we will see that this is not possible if a first Novikov homology group is non-zero.

Let G be a group and $C \to \mathbb{Z}$ be a free resolution over $\mathbb{Z}G$ with C_i finitely generated for $i \leq k$. By Bieri and Renz [5, Lm.3.1] there is a subcomplex $C^+ \subset C$ of $\mathbb{Z}G_\chi$-modules which is finitely generated free for $i \leq k$ and such that the rank of C_i^+ equals the rank of C_i. Given a $g \in G$ with $\chi(g) > 0$ we can now form an inverse system $H_*(C/g^i C^+) \longleftarrow H_*(C/g^{i+j} C^+)$ with $i, j \geq 0$.

Lemma 4.1. *Let G and C be as above. If $\chi \in \Sigma^{k-1}(G;\mathbb{Z}) - \Sigma^k(G;\mathbb{Z})$, then the inverse limit $\varprojlim H_k(C/g^iC^+)$ is non-trivial, where $g \in G$ satisfies $\chi(g) > 0$.*

Proof. By [5, Thm.3.2] we get that for every i there is an $x_i \in \tilde{H}_{k-1}(g^iC^+)$ with $0 \neq j_*x_i \in \tilde{H}_{k-1}(C^+)$. But by the Sigma criterion [5, Thm.C(III)], x_i can be represented in any $\tilde{H}_{k-1}(g^mC^+)$ with $m \geq i$. Therefore there is a non-trivial element in $\varprojlim \tilde{H}_{k-1}(g^iC^+) \cong \varprojlim H_k(C/g^iC^+)$. □

Lemma 4.2. *In the situation of Lemma 4.1 there is a surjective homomorphism*

$$\mathrm{Tor}_k^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) \longrightarrow \varprojlim H_k(C/g^iC^+)$$

of abelian groups.

Proof. For $g \in G$ with $\chi(g) > 0$ there is an inverse system of abelian groups $\mathbb{Z}G/g^i\mathbb{Z}G_\chi \longleftarrow \mathbb{Z}G/g^{j+i}\mathbb{Z}G_\chi$ whose inverse limit is exactly $\widehat{\mathbb{Z}G}_\chi$. If C_l is a finitely generated free $\mathbb{Z}G$-module, then $\varprojlim C_l/g^iC_l^+ = \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G} C_l$. We get the standard short exact sequence

$$0 \longrightarrow \varprojlim{}^1 H_{l+1}(C/g^iC^+) \longrightarrow H_l(\varprojlim C/g^iC^+) \longrightarrow \varprojlim H_l(C/g^iC^+) \longrightarrow 0$$

and as we have that C_l is finitely generated for $l \leq k$ we get a surjective homomorphism $\mathrm{Tor}_k^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) \to \varprojlim H_k(C/g^iC^+)$. Notice that there is only an inclusion $\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G} C_{k+1} \to \varprojlim C_{k+1}/g^iC_{k+1}^+$ in the case that C_{k+1} is not finitely generated, but we still get a surjection $\mathrm{Tor}_k^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) \to H_k(\varprojlim C/g^iC^+)$. The Lemma follows. □

As $\varprojlim H_k(C/g^iC^+) \neq 0$ by Lemma 4.1, we get a non-trivial homomorphism $\mathrm{Tor}_k^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) \to H_k(C/C^+)$ of abelian groups.

Proposition 4.3. *For $i = 1, 2$ let G_i be a group of type FP_{k_i} for some $k_i \geq 1$, and $\chi_i : G_i \to \mathbb{R}$ a nonzero homomorphism with $\chi_i \in \Sigma^{k_i-1}(G_i;\mathbb{Z}) - \Sigma^{k_i}(G_i;\mathbb{Z})$. Assume that there is a non-trivial abelian group homomorphism $\varphi_1 : \mathrm{Tor}_{k_1}^{\mathbb{Z}G_1}(\widehat{\mathbb{Z}G_1}_{\chi_1}, \mathbb{Z}) \to \mathbb{Z}$ which factors through $H_{k_1}(C/C^+)$. Then*

$$\begin{aligned} \mathrm{Tor}_i^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) &= 0 \qquad \text{for } i \leq k_1 + k_2 - 1 \\ \mathrm{Tor}_{k_1+k_2}^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) &\neq 0 \end{aligned}$$

where $G = G_1 \times G_2$ and $\chi : G \to \mathbb{R}$ is the sum of χ_1 and χ_2.

Proof. It follows from Corollary 3.3 that $\mathrm{Tor}_i^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) = 0$ for $i \leq k$, so we need to show that $\mathrm{Tor}_{k+1}^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) \neq 0$.

By assumption $\varphi_1 : \mathrm{Tor}_{k_1}^{\mathbb{Z}G_1}(\widehat{\mathbb{Z}G_1}_{\chi_1}, \mathbb{Z}) \to \mathbb{Z}$ factors through $H_{k_1}(C/C^+)$. Denote the image in $H_{k_1}(C/C^+)$ by A_1. Furthermore, there is a non-trivial homomorphism $\varphi_2 : \mathrm{Tor}_k^{\mathbb{Z}G_2}(\widehat{\mathbb{Z}G_2}_{\chi_2}, \mathbb{Z}) \to H_k(D/D^+)$, where $D \to \mathbb{Z}$ is a free resolution over $\mathbb{Z}G_2$. Let us denote the image by A_2. These homomorphisms combine to an epimorphism

$$\varphi : \mathrm{Tor}_{k_1}^{\mathbb{Z}G_1}(\widehat{\mathbb{Z}G_1}_{\chi_1}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Tor}_{k_2}^{\mathbb{Z}G_2}(\widehat{\mathbb{Z}G_2}_{\chi_2}, \mathbb{Z}) \to A_1 \otimes A_2$$

of abelian groups.

Given $x \in \operatorname{Tor}_{k_1}^{\mathbb{Z}G_1}(\widehat{\mathbb{Z}G_1}_{\chi_1}, \mathbb{Z})$ and $y \in \operatorname{Tor}_{k_2}^{\mathbb{Z}G_2}(\widehat{\mathbb{Z}G_2}_{\chi_2}, \mathbb{Z})$, there exists an $M > 0$ such that

$$\varphi(g_1 x \otimes g_2 y) = 0, \text{ if } \chi_1(g_1) + \chi_2(g_2) \geq M \tag{3}$$

To see this notice that there is $M > 0$ such that $\varphi_1(g_1 x) = 0 \in H_{k_1}(C/C^+)$ if $\chi_1(g) \geq M/2$ as the support of a cycle representing x will be contained in N if $\chi_1(g_1)$ is big enough. Here notice that $C/C^+ \cong \left(\widehat{\mathbb{Z}G_1}_{\chi_1} \otimes_{\mathbb{Z}G_1} C\right) / \left(\widehat{\mathbb{Z}G_1}_{\chi_1}^+ \otimes_{\mathbb{Z}G_{1\chi_1}} C^+\right)$, where

$$\widehat{\mathbb{Z}G_1}_{\chi_1}^+ \quad = \quad \{\lambda \in \widehat{\mathbb{Z}G_1}_{\chi_1} \mid \operatorname{supp} \lambda \subset \chi_1^{-1}([0,\infty))\}.$$

A similar argument gives $\varphi_2(g_2 y) = 0 \in H_k(D/D^+)$ for $\chi_2(g_2) \geq M/2$ big enough. Let us denote

$$\begin{aligned} \Lambda &= \widehat{\mathbb{Z}G}_\chi \\ R &= \widehat{\mathbb{Z}G_1}_{\chi_1} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}G_2}_{\chi_2} \end{aligned}$$

If $\lambda \in \Lambda$, then there is a $\bar{\lambda} \in \mathbb{Z}G$ with $\operatorname{supp} \lambda - \bar{\lambda} \subset \chi^{-1}([M,\infty))$ for any $M > 0$. Now define a homomorphism of abelian groups

$$\psi : \Lambda \otimes_R \left(\operatorname{Tor}_{k_1}^{\mathbb{Z}G_1}(\widehat{\mathbb{Z}G_1}_{\chi_1}, \mathbb{Z}) \otimes_{\mathbb{Z}} \operatorname{Tor}_{k_2}^{\mathbb{Z}G_2}(\widehat{\mathbb{Z}G_2}_{\chi_2}, \mathbb{Z})\right) \to A_1 \otimes A_2$$

by $\psi(\lambda \otimes_R (x \otimes y)) = \varphi(\bar{\lambda} \cdot (x \otimes y))$, where $\operatorname{supp} \lambda - \bar{\lambda} \subset \chi^{-1}([M,\infty))$ with $M > 0$ as in (3). This does not depend on $\bar{\lambda}$, because if $\tilde{\lambda} \in \mathbb{Z}G$ also satisfies $\operatorname{supp} \lambda - \tilde{\lambda} \subset \chi^{-1}([M,\infty))$, we get $\operatorname{supp} \bar{\lambda} - \tilde{\lambda} \subset \chi^{-1}([M,\infty))$ and

$$\varphi(\bar{\lambda} \cdot (x \otimes y)) \quad = \quad \varphi(\tilde{\lambda} \cdot (x \otimes y))$$

by (3). Let us show that ψ is well defined, that is, for $r \in R$ let us show that

$$\psi(\lambda \otimes_R r \cdot (x \otimes y)) = \psi(\lambda \cdot r \otimes_R (x \otimes y)).$$

There is a $K \leq 0$ with $\operatorname{supp} r \cup \operatorname{supp} \lambda \subset \chi^{-1}([K,\infty))$. Let $M > 0$ satisfy

$$\varphi(h_1 x \otimes h_2 y) = 0 \text{ for } \chi(h_1, h_2) \geq M.$$

Then

$$\varphi((g_1, g_2) \cdot r \cdot (x \otimes y)) = 0 \text{ for } \chi(g_1, g_2) \geq M - K.$$

Write $\lambda = \bar{\lambda} + \mu$ with $\bar{\lambda} \in \mathbb{Z}G$ and $\operatorname{supp} \mu \subset \chi^{-1}([M-K,\infty))$. Also let $r = \bar{r} + \nu$ with $\bar{r} \in \mathbb{Z}G$ and $\operatorname{supp} \nu \subset \chi^{-1}([M-K,\infty))$. Notice that $\nu \in R$. Then $\lambda \cdot r = \bar{\lambda}\bar{r} + \bar{\lambda}\nu + \mu\bar{r} + \mu\nu$ and $\operatorname{supp} \lambda r - \bar{\lambda}\bar{r} \subset \chi^{-1}([M,\infty))$.
According to the definition of ψ we get

$$\begin{aligned} \psi(\lambda \otimes r(x \otimes y)) &= \varphi(\bar{\lambda} \cdot r(x \otimes y)) \\ &= \varphi(\bar{\lambda}\bar{r}(x \otimes y) + \bar{\lambda}\nu(x \otimes y)) \\ &= \varphi(\bar{\lambda}\bar{r}(x \otimes y)) \\ &= \psi(\lambda r \otimes (x \otimes y)) \end{aligned}$$

which shows that ψ is indeed a well defined homomorphism of groups. Since φ is surjective, the same holds for ψ. As there is a non-trivial homomorphism $A_1 \to \mathbb{Z}$, it follows that $A_1 \otimes A_2$ is non-trivial. By Corollary 3.3 this shows that $\operatorname{Tor}_{k_1+k_2}^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) \neq 0$. □

The assumptions of Proposition 4.3 are not always satisfied, as Example 3.7 shows, but for $k_1 = 1$ they are satisfied:

Theorem 4.4. *Let G_1 and G_2 be groups of type FP_n. Assume $\chi_1 : G_1 \to \mathbb{R}$ is a nonzero homomorphism with $\chi_1 \notin \Sigma^1(G_1;\mathbb{Z})$. If $\chi_2 \notin \Sigma^{n-1}(G_2;\mathbb{Z})$, then $\chi = \chi_1 + \chi_2 \notin \Sigma^n(G_1 \times G_2;\mathbb{Z})$.*

Proof. We have to show that there is always a non-trivial homomorphism

$$\mathrm{Tor}_1^{\mathbb{Z}G_1}(\widehat{\mathbb{Z}G_1}_{\chi_1}, \mathbb{Z}) \to \mathbb{Z}$$

which factors through $H_1(C/C^+)$. But in degree 1 we can look at $H_1(X;\widehat{\mathbb{Z}G_1}_{\chi_1}) \to H_1(\tilde{X}, N) \to \mathbb{Z}$, where X is a connected CW-complex with $\pi_1(X) = G_1$ and finite 1-skeleton, and where $N = f^{-1}([0,\infty))$ for a map $f : \tilde{X} \to \mathbb{R}$ with $f(gx) = \chi_1(g) + f(x)$ for all $x \in \tilde{X}$ and $g \in G_1$.

We can assume that X has only one 0-cell x and a 1-cell for every generator of G_1 from a finite generating set.

Let $\sigma \subset \tilde{X}$ be a path in the 1-skeleton of $\tilde{X}$ which represents a nonzero element $[\sigma] \in H_1(\tilde{X}, N)$. We can think of σ as a 1-chain in $\tilde{X}$ with $\partial\sigma = v - hv$, where $v \in N$ is a lift of x and $h \in G_1$.

Let $g \in G_1$ be a generator that satisfies $\chi_1(g) > 0$ and let $e \subset \tilde{X}$ be a 1-cell connecting v and gv. We can assume that $e \subset N$ and think of e as a 1-chain with $\partial e = v - gv$. Then

$$z \;=\; \sigma - (1-g)^{-1}e + h(1-g)^{-1}e$$

represents a cycle in $\widehat{\mathbb{Z}G_1}_{\chi_1} \otimes_{\mathbb{Z}G_1} C_1(\tilde{X})$ whose Novikov homology class is send to $[\sigma]$ under $\varphi_1 : H_1(X;\widehat{\mathbb{Z}G_1}_{\chi_1}) \to H_1(\tilde{X}, N)$. Now $H_1(\tilde{X}, N) \cong \tilde{H}_0(N)$ is a free abelian group and $\tilde{H}_0(C(v) \cup C(hv)) \cong \mathbb{Z}$ is a direct summand, where $C(v)$ and $C(hv)$ are the connected components of v and hv in N, respectively.

This gives the necessary surjection $H_1(X;\widehat{\mathbb{Z}G_1}_{\chi_1}) \to \mathbb{Z}$ so the result follows from Proposition 4.3 in connection with Lemma 2.2. □

Theorem 4.5. *Let G_1 and G_2 be groups of type FP_n with $n \leq 3$. Then $\chi \notin \Sigma^n(G_1 \times G_2;\mathbb{Z})$ if and only if $\chi_1 \notin \Sigma^p(G_1;\mathbb{Z})$ and $\chi_2 \notin \Sigma^q(G_2;\mathbb{Z})$ for some p and q with $p + q = n$.*

Proof. Let $\chi_1 \notin \Sigma^p(G_1;\mathbb{Z})$ and $\chi_2 \notin \Sigma^q(G_2;\mathbb{Z})$ for some p and q with $p + q = n$. As $n \leq 3$, we get $p \leq 1$ or $q \leq 1$. Without loss of generality we can assume $p \leq 1$. If $p = 1$ we get $\chi \notin \Sigma^{r+1}(G;\mathbb{Z})$ by Theorem 4.4, where $r \leq q$ is such that $\chi_2 \in \Sigma^{r-1}(G_2;\mathbb{Z}) - \Sigma^r(G_2;\mathbb{Z})$. As $r + 1 \leq q + p \leq n$, the result follows. If $p = 0$, note that $\chi_1 = 0$ and

$$\mathrm{Tor}_0^{\mathbb{Z}G_1}(\widehat{\mathbb{Z}G_1}_{\chi_1}, \mathbb{Z}) \;=\; \mathbb{Z}$$

and so we get

$$\mathrm{Tor}_r^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) \;\cong\; \widehat{\mathbb{Z}G}_\chi \otimes_R \mathrm{Tor}_r^{\mathbb{Z}G_2}(\widehat{\mathbb{Z}G_2}_{\chi_2}, \mathbb{Z})$$

where $r \leq q$ satisfies $\chi_2 \in \Sigma^{r-1}(G_2;\mathbb{Z}) - \Sigma^r(G_2;\mathbb{Z})$ and $R = \mathbb{Z}G_1 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}G_2}_{\chi_2}$. By a similar argument as in the proof of Proposition 4.3 we see that this group is non-trivial. □

5. Applications to right-angled Artin groups

In this section we want to look for right-angled Artin groups which do have a non-trivial homomorphism of abelian groups $\varphi_1 : \mathrm{Tor}_k^{\mathbb{Z}G}(\widehat{\mathbb{Z}G_\chi}, \mathbb{Z}) \to \mathbb{Z}$ which factors through $H_k(C/C^+)$, so that Proposition 4.3 does apply. In the case that $G = G_L$ is a right-angled Artin group we have to look for a non-trivial homomorphism

$$H_k(X_L; \widehat{\mathbb{Z}G_{L\chi}}) \to H_k(\tilde{X}_L, N) \to \mathbb{Z}$$

where X_L is the standard $K(G_L, 1)$ as in [1].
Let $\tilde{X} = \tilde{X}_L$ be the universal cover of X_L and for a given homomorphism $\chi : G_L \to \mathbb{R}$ define $h : \tilde{X} \to \mathbb{R}$ as in Bux and Gonzalez [6], that is, h is linear on cells. If $J \subset \mathbb{R}$ is an interval, denote $\tilde{X}_J = h^{-1}(J)$ and $\tilde{X}_J^0$ the union of 0-cells in $\tilde{X}_J$.

Proposition 5.1. *Let L be a finite flag complex and G_L the corresponding right-angled Artin group. Let $\chi : G_L \to \mathbb{R}$ be a homomorphism with $\chi(v_i) \neq 0$ for all generators v_i of G_L. Then there exists a surjection*

$$H_k(X_L; \widehat{\mathbb{Z}G_{L\chi}}) \longrightarrow \tilde{H}_{k-1}(L)$$

which factors through $H_k(\tilde{X}, \tilde{X}_{[0,\infty)})$ for all integers k.

Proof. In the case when χ is equal to 1 on every generator, this follows directly from [1, Thm.7.1]. In general we have to combine the methods of [6, Sec.1] and [1, Sec.6,7] to get that

$$H_*(\tilde{X}, \tilde{X}_{[t,\infty)}) \quad \cong \bigoplus_{v \in \tilde{X}^0_{(-\infty,0)}} \tilde{H}_{*-1}(L)$$

and that for $t < t'$ the map $H_*(\tilde{X}, \tilde{X}_{[t,\infty)}) \leftarrow H_*(\tilde{X}, \tilde{X}_{[t',\infty)})$ is the obvious projection. We omit the details. Combining with Lemma 3.6 we get the result. □

Theorem 5.2. *Let k be an integer, and L a finite flag complex with $\tilde{H}_i(L) = 0$ for $i < k$ and $\tilde{H}_k(L)$ contains $\mathbb{Z}$ as a direct summand. Let $\chi_1 : G_L \to \mathbb{R}$ be a homomorphism with $\chi_1(v_i) \neq 0$ for all generators v_i of G_L. Let G be a group of type FP_{n+1} and $\chi_2 : G \to \mathbb{R}$ satisfies $\chi_2 \in \Sigma^{n-1-k}(G; \mathbb{Z}) - \Sigma^{n-k}(G; \mathbb{Z})$. Then $\chi = \chi_1 + \chi_2$ satisfies $\chi \in \Sigma^n(G_L \times G; \mathbb{Z}) - \Sigma^{n+1}(G_L \times G; \mathbb{Z})$.*

Proof. We have $\chi_1 \in \Sigma^k(G_L; \mathbb{Z}) - \Sigma^{k+1}(G_L; \mathbb{Z})$ and by Proposition 5.1 we can use Propositon 4.3 which implies that $\chi \in \Sigma^n(G_L \times G; \mathbb{Z}) - \Sigma^{n+1}(G_L \times G; \mathbb{Z})$. □

So with the conditions described in Theorem 5.2 on G_L and χ_1 we get that the product formula holds for χ_1 and every G and $\chi_2 : G \to \mathbb{R}$. Note that $\tilde{H}_k(L)$ is a finitely generated abelian group, so if this group does not contain a direct summand $\mathbb{Z}$, we can always find another group G (in fact a right-angled Artin group) such that the product formula will not hold.

Remark 5.3. In order to drop the condition that $\chi(v_i) \neq 0$ for all generators v_i of G_L, one has to replace the reduced homology of L by the reduced homology of certain ascending links described in [6]. We will avoid stating a more general version of Theorem 5.2, but note that such a theorem has to consider not only the homology of L, but also the homology of certain subcomplexes of L. We refer the reader to [6] for details on the subcomplexes involved.

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