HOMOLOGY OF PLANAR POLYGON SPACES

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ABSTRACT. In this paper we study topology of the variety of closed planar n-gons with given side lengths $l_1, \dots, l_n$. The moduli space M_ℓ where $\ell = (l_1, \dots, l_n)$, encodes the shapes of all such n-gons. We describe the Betti numbers of the moduli spaces M_ℓ as functions of the length vector $\ell = (l_1, \dots, l_n)$. We also find sharp upper bounds on the sum of Betti numbers of M_ℓ depending only on the number of links n. Our method is based on an observation of a remarkable interaction between Morse functions and involutions under the condition that the fixed points of the involution coincide with the critical points of the Morse function.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Given a string $\ell = (l_1, \dots, l_n)$ of n positive real numbers $l_i > 0$ one considers the moduli space M_ℓ of closed planar polygonal curves having side lengths l_i. Points of M_ℓ parametrize different shapes of such polygons. Formally M_ℓ is defined as the factor space

$$M_\ell = \{(u_1, \dots, u_n) \in S^1 \times \dots \times S^1; \sum_{i=1}^n l_i u_i = 0 \in \mathbf{C}\} / \mathrm{SO}(2).$$

Here $u_i \in S^1 \subset \mathbf{C}$ denote the unit vectors in the directions of the sides of a polygon; the group of rotations $SO(2)$ acts diagonally on $(u_1, \dots, u_n)$.

Viewed differently, M_ℓ is the configuration space of a planar linkage, a planar mechanism consisting of n bars of length $l_1, \dots, l_n$ connected by revolving joints. Such mechanisms play an important role in robotics where they describe closed kinematic chains and are used widely as elementary parts of more complicated mechanisms. Knowing the topology of M_ℓ (for different vectors ℓ) can be used in designing control programmes and motion planning algorithms for mechanisms.

The length vector ℓ is called *generic* if $\sum_{i=1}^n l_i \epsilon_i \neq 0$ for any choice $\epsilon_i = \pm 1$. It is known that for a generic length vector ℓ the space M_ℓ is a closed smooth manifold of dimension $n-3$. If the length vector ℓ is not generic then M_ℓ is a compact $(n-3)$-dimensional manifold with finitely many singular points.

2000 *Mathematics Subject Classification.* Primary 58Exx.

Key words and phrases. Polygon spaces, Morse theory of manifolds with involutions, varieties of linkages.

M. Farber was supported by the Royal Society Wolfson Research Merit Award and by EPSRC grant "Applications of Algebraic Topology".

The moduli spaces M_ℓ of planar polygonal linkages were studied extensively by many mathematicians; we will mention W. Thurston and J. Weeks [12], K. Walker [14], A. A. Klyachko [8], M. Kapovich and J. Millson [6], J.-Cl. Hausmann and A. Knutson [2] and others.

Our goal in this paper is to give a general formula for the Betti numbers of the moduli space M_ℓ as functions of the length vector ℓ. Our results cover both generic and non-generic vectors ℓ. In the case of generic ℓ the Betti numbers of M_ℓ can easily be extracted from the results of the unpublished thesis of K. Walker [14].

Formulae for Betti numbers of polygon spaces in three-dimensional space are known (see papers of A.A. Klyachko [8] and J.-Cl. Hausmann and A. Knutson [2]). Also, J.-Cl. Hausmann and A. Knutson describe cohomology with $\mathbf{Z}_2$ coefficients of the factor $\bar{M}_\ell$ of M_ℓ with respect to the natural involution, see [2], Theorem 9.1.

A. Klyachko in his beautiful work [8] uses a remarkable symplectic structure on the moduli space of linkages in $\mathbf{R}^3$ in an essential way. His technique is based on properties of Hamiltonian circle actions (the perfectness of the Hamiltonian viewed as a Morse function). The subsequent important paper of J.-Cl. Hausmann and A. Knutson employs methods of symplectic topology as well: they apply the method of symplectic reduction. Note that J.-Cl. Hausmann and A. Knutson go one step further and compute the multiplicative structure on cohomology, however their description is not very explicit as it uses the language of generators and relations. Symplectic methods play also a central role in the work of M. Kapovich and J. Millson [7].

The moduli spaces of planar linkages M_ℓ do not carry symplectic structures in general. Therefore methods of symplectic topology are not applicable in this problem.

The proof of our main result (see Theorem 1 below) is obtained in a very simple manner, it uses a remarkable interaction between Morse functions and involutions under the condition that fixed points of the involution coincide with the critical points of the Morse function.

To state our main theorem we need the following definitions. A subset $J \subset \{1, \dots, n\}$ is called *short* if

$$\sum_{i \in J} l_i < \sum_{i \notin J} l_i.$$

The complement of a short subset is called *long*. A subset $J \subset \{1, \dots, n\}$ is called *median* if

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i.$$

Clearly, median subsets exist only if the length vector ℓ is not generic. Note the following simple observation: any two subsets $J, J' \subset \{1, \dots, n\}$ have a nonempty intersection $J \cap J' \neq \emptyset$ provided that one of the subsets is long and the other is either long or median.

Theorem 1. *Fix a link of the maximal length l_i, i.e. such that $l_i \geq l_j$ for any $j = 1, 2, \ldots, n$. For every $k = 0, 1, \ldots, n-3$ denote by a_k and b_k correspondingly the number of short and median subsets of $\{1, \ldots, n\}$ of cardinality $k+1$ containing i. Then the homology group $H_k(M_\ell; \mathbf{Z})$ is free abelian of rank*

$$a_k + b_k + a_{n-3-k}, \tag{1}$$

for any $k = 0, 1, \ldots, n-3$.

By Theorem 1 the Poincaré polynomial

$$p(t) = \sum_{k=0}^{n-3} \dim H_k(M_\ell; \mathbf{Q}) \cdot t^k$$

of M_ℓ can be written in the form

$$q(t) + t^{n-3} q(t^{-1}) + r(t) \tag{2}$$

where

$$q(t) = \sum_{k=0}^{n-3} a_k t^k, \quad r(t) = \sum_{k=0}^{n-3} b_k t^k; \tag{3}$$

the numbers a_k and b_k are described in the statement of Theorem 1.

A proof of Theorem 1 is given below in §5. In the rest of this introduction we illustrate the statement of Theorem 1 by several examples.

Example 1. Suppose that $n = 5$ and $l_1 = 3$, $l_2 = 2$, $l_3 = 2$, $l_4 = 1, l_5 = 1$. Then $l_1 = 3$ is the longest link and short subsets of $\{1, \ldots, 5\}$ containing 1 are $\{1\}$, $\{1, 4\}$ and $\{1, 5\}$. Hence $a_0 = 1$, $a_1 = 2$ and by Theorem 1 the Poincaŕe polynomial of M_ℓ equals $1 + 4t + t^2$. We conclude that M_ℓ is a closed orientable surface of genus 2.

Example 2. Consider the zero-dimensional Betti number

$$a_0 + b_0 + a_{n-3}$$

of M_ℓ as given by Theorem 1. We want to show that this number can take values $0, 1, 2$; the first possibility is clearly equivalent to $M_\ell = \emptyset$. Without loss of generality we may assume that $l_1 \leq l_2 \leq \cdots \leq l_n$. If $\{n\}$ is short then $a_0 = 1$ and $b_0 = 0$. If $\{n\}$ is median then $a_0 = 0$ and $b_0 = 1$; in this case clearly M_ℓ is a single point. If $\{n\}$ is long then $a_k = 0 = b_k$ for any k and hence $M_\ell = \emptyset$. We obtain that $M_\ell = \emptyset$ if and only if there are no long one-element subsets of $\{1, \ldots, n\}$ – a result first established by Kapovich and Millson in [6].

Let us show that the number a_{n-3} equals 0 or 1. Clearly, a_{n-3} coincides with the number of long two-element subsets $\{r, s\} \subset \{1, \ldots, n-1\}$. There may exist at most one such pair: if $\{r', s'\}$ is another long pair with $r \neq r'$, $r \neq s'$, then $\{r, n\}$ and $\{r', s'\}$ would be two disjoint long subsets which is impossible. We obtain that $a_{n-3} = 1$ if and only if the pair $\{n-2, n-1\}$ is long and $a_{n-3} = 0$ otherwise.

We see that the moduli space M_ℓ has two connected components if and only if the set $\{n-2, n-1\}$ is long. In this case the length vector ℓ must be generic and short subsets $J \subset \{1, \dots, n\}$ containing n are exactly the subsets containing neither $n-2$ nor $n-1$. We see that the Poincaré polynomial of M_ℓ in this case equals $2(1+t)^{n-3}$. M. Kapovich and J. Millson [6] showed that if M_ℓ is disconnected then it is diffeomorphic to the disjoint union of two copies of the torus T^{n-3}.

Example 3. As another example consider the equilateral case when $l_j = 1$ for all j. Assume first that $n = 2r+1$ is odd and hence ℓ is generic. The short subsets in this case are subsets of $\{1, \dots, n\}$ of cardinality $\le r$. We may fix the index $\{n\}$ as representing the longest link. Hence we find that $b_k = 0$ vanishes and a_k equals

$$a_k = \begin{cases} \binom{n-1}{k} & \text{for } k \le r-1, \\ 0, & \text{for } k \ge r. \end{cases} \tag{4}$$

By Theorem 1 the Betti numbers of M_ℓ are given by

$$b_k(M_\ell) = \begin{cases} \binom{n-1}{k} & \text{for } k < r-1, \\ 2 \cdot \binom{n-1}{r-1} & \text{for } k = r-1, \\ \binom{n-1}{k+2} & \text{for } k > r-1. \end{cases} \tag{5}$$

Note that the sum of Betti numbers in this example equals

$$\sum_{k=0}^{n-3} b_k(M_\ell) = 2^{n-1} - \binom{n-1}{r}, \quad \text{where} \quad n = 2r+1. \tag{6}$$

Example 4. Consider now the equilateral case $l_j = 1$ with n is even, $n = 2r+2$. The length vector is now not generic. The short subsets are all subsets of cardinality $\le r$ and the median subsets are all subsets of cardinality $r+1$. Hence we find that $b_k = 0$ for $k \ne r$ and

$$b_r = \binom{2r+1}{r} \tag{7}$$

and the numbers a_k are given by formula (4). Applying Theorem 1 we find

$$b_k(M_\ell) = \begin{cases} \binom{n-1}{k} & \text{for } k \le r-1, \\ \\ \binom{n}{r} & \text{for } k = r, \\ \\ \binom{n-1}{k+2} & \text{for } r+1 \le k \le n-3. \end{cases} \tag{8}$$

The sum of Betti numbers in this example is

$$\sum_{k=0}^{n-3} b_k(M_\ell) \; = \; 2^{n-1} - \binom{n-1}{r}, \quad \text{where} \quad n = 2r+2. \tag{9}$$

The results described in Examples 3 and 4 were obtained earlier in [4], [5] by different methods.

In the next section we shall see that Examples 3 and 4 give moduli spaces M_ℓ with the maximal possible total Betti number for all length vectors ℓ having the given number of links n.

We are pleased to have the opportunity to thank the anonymous referee for his helpful comments.

2. Maximum of the total Betti number of M_ℓ

It is well known that the moduli space of pentagons M_ℓ with a generic length vector $\ell = (l_1, \dots, l_5)$ is a compact orientable surface of genus not exceeding 4, see [9]. In the equilateral case, i.e. if $\ell = (1,1,1,1,1)$, M_ℓ is indeed an orientable surface of genus 4 (it is a special case of (6)) and hence the above upper bound for pentagons is sharp. In this section we state a theorem generalizing this result for arbitrary n. Namely, we prove that for any length vector $\ell = (l_1, \dots, l_n)$ the sum of the Betti numbers

$$\sum_{i=0}^{n-3} b_i(M_\ell) \tag{10}$$

is less or equal than the sum of Betti numbers of the moduli space of the equilateral linkage with the same number of sides n.

Theorem 2. *Let $\ell = (l_1, \dots, l_n)$ be a length vector, $l_i > 0$. Denote by r the number $[\frac{n-1}{2}]$. Then the sum of Betti numbers of the moduli space M_ℓ does not exceed*

$$B_n = 2^{n-1} - \binom{n-1}{r}. \tag{11}$$

This estimate is sharp: B_n equals the sum of Betti numbers of the moduli space of planar equilateral n-gons, see (6), (9).

Note that for n even the equilateral linkage with n sides is not generic and hence Theorem 2 does not answer the question about the maximum of the total Betti number on the set of all generic length vectors with n even.

Theorem 3. *Assume that n is even and $\ell = (l_1, \dots, l_n)$ is a generic length vector. Then the sum of Betti numbers of M_ℓ does not exceed*

$$B'_n \,=\, 2 \cdot B_{n-1}, \tag{12}$$

where B_k is defined by (11). This upper bound is achieved on the length vector $\ell = (1, 1, \dots, 1, \epsilon)$ where $0 < \epsilon < 1$ and the number of ones is $n - 1$.

Note that $M_{(1,\dots,1,\epsilon)}$ is diffeomorphic to the product $M_{(1,\dots,1)} \times S^1$ (the number of ones in both cases equals $2r + 1$), see Prop. 6.1 of [3]. Hence the sum of Betti numbers of $M_{(1,\dots,1,\epsilon)}$ is twice the sum of Betti numbers of $M_{(1,\dots,1)}$.

Proofs of Theorems 2 and 3 are given below in section §6.

The asymptotic behavior of B_n (given by (11)) can be recovered using available information about Catalan numbers

$$C_r = \frac{1}{r+1} \cdot \left(\begin{array}{c} 2r \\ r \end{array} \right) \sim \frac{2^{2r}}{\sqrt{\pi} r^{3/2}},$$

see [13]. One obtains the following asymptotic formula

$$B_n \sim 2^{n-1} \cdot \left(1 - \sqrt{\frac{2}{n\pi}} \right) \tag{13}$$

which is valid for even and odd n.

From the discussion of Example 2 we know that in the case when M_ℓ is disconnected the sum of Betti numbers of M_ℓ equals 2^{n-2} which is approximately half of B_n, see (13).

3. Morse theory on manifolds with involutions

Our main tool in computing the Betti numbers of the moduli space of planar polygons M_ℓ is Morse theory of manifolds with involution.

Theorem 4. *Let M be a smooth compact manifold with boundary. Assume that M is equipped with a Morse function $f : M \to [0, 1]$ and with a smooth involution $\tau : M \to M$ satisfying the following properties:*

(1) f is τ-invariant, i.e. $f(\tau x) = f(x)$ for any $x \in M$;
(2) The critical points of f coincide with the fixed points of the involution;
(3) $f^{-1}(1) = \partial M$ and $1 \in [0, 1]$ is a regular value of f.

Then each homology group $H_i(M; \mathbf{Z})$ is free abelian of rank equal the number of critical points of f having Morse index i. Moreover, the induced map

$$\tau_* : H_i(M; \mathbf{Z}) \to H_i(M; \mathbf{Z})$$

coincides with multiplication by $(-1)^i$ for any i.

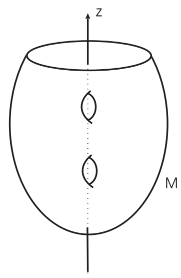

FIGURE 1. Surface in $\mathbf{R}^3$

As an illustration for Theorem 4 consider a surface is $\mathbf{R}^3$ (see Figure 1) which is symmetric with respect to the z-axis. The function f is the orthogonal projection onto the z-axis, the involution $\tau : M \to M$ is given by $\tau(x, y, z) = (-x, -y, z)$.

The critical points of f are exactly the intersection points of M with the z-axis.

Proof of Theorem 4. Choose a Riemannian metric on M which is invariant with respect to τ.

Let $p \in M$ be a critical point of f. By our assumption, p must be a fixed point of τ, i.e. $\tau(p) = p$. We claim that the differential of τ at p is multiplication by -1, i.e.

$$d\tau_p(v) = -v, \quad \text{for any} \quad v \in T_pM. \tag{14}$$

Firstly, since τ is an involution, $d\tau_p$ must have eigenvalues ± 1. Assume that there exists a vector $v \in T_pM$ with $d\tau_p(v) = v$. Then the geodesic curve starting from p in the direction of v is invariant with respect to τ implying that p is not isolated in the fixed point set of τ. This contradicts our assumption and hence $d\tau_p$ must have eigenvalue -1 only. This proves (14).

Consider the gradient vector field v of f with respect to the Riemannian metric. We will assume that v satisfies the transversality condition, i.e. all stable and unstable manifolds of the critical points intersect transversally. To show that such a vector field exists, one may start with an arbitrary τ-invariant vector field and apply the technique of Milnor [11]. In the proof of Theorem 5.2 from [11] the vector field is only changed in a cylindrical neighborhood of a codimension 1 submanifold. In our situation τ acts freely on such a neighborhood; hence applying the argument to the quotient space, one obtains a τ-invariant vector field satisfying the transversality condition.

The vector field v is τ-invariant which means that

$$v_{\tau(x)} = d\tau_x(v_x), \quad x \in M. \tag{15}$$

The Morse - Smale chain complex $(C_*(f), \partial)$ of f has the critical points of f as its basis and the differential is given by

$$\partial(p) = \sum_q [p:q]\, q \tag{16}$$

where in the sum q runs over the critical points q with Morse index $\mathrm{ind}(q) = \mathrm{ind}(p) - 1$. The incidence numbers $[p:q] \in \mathbf{Z}$ are defined as follows

$$[p:q] = \sum_\gamma \epsilon(\gamma), \quad \epsilon(\gamma) = \pm 1, \tag{17}$$

where $\gamma : (-\infty, \infty) \to M$ are trajectories of the negative gradient flow $\gamma'(t) = -v_{\gamma(t)}$ satisfying the boundary conditions $\gamma(t) \to p$ as $t \to -\infty$ and $\gamma(t) \to q$ as $t \to +\infty$.

Observe that if γ is a trajectory as above then $\tau \circ \gamma$ is another such trajectory. Indeed, using (15) we find $(\tau \circ \gamma)' = d\tau(\gamma') = -d\tau(v_{\gamma(t)}) = -v_{\tau(\gamma(t))}$.

Theorem 4 would follow once we show that

$$\epsilon(\gamma) + \epsilon(\tau \circ \gamma) = 0, \tag{18}$$

i.e. the total contribution into (17) of a pair of symmetric trajectories is zero. Hence all incidence coefficients vanish $[p:q] = 0$ and the differentials of the Morse - Smale complex are trivial.

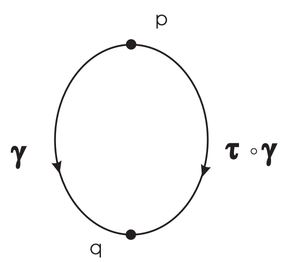

FIGURE 2. Two symmetric trajectories of the negative gradient flow

To prove (18) we first recall the definition of the sign $\epsilon(\gamma) \in \{1, -1\}$, see [11]. For a critical point p of f we denote by $W^u(p)$ and $W^s(p)$ the unstable and stable manifolds of p. Recall that $W^u(p)$ is the union of the trajectories $\gamma : (-\infty, \infty) \to M$ satisfying the differential equation $\gamma'(t) = -v_{\gamma(t)}$ and the boundary condition $\gamma(t) \to p$ as $t \to -\infty$. The stable manifold $W^s(p)$ is defined similarly but the boundary condition in this case becomes $\gamma(t) \to p$ as $t \to +\infty$.

Fix an orientation of the stable manifold $W^s(p)$ for every critical point $p \in M$. Since $W^s(p)$ and $W^u(p)$ are of complementary dimension and intersect transversally at p, the orientation of $W^s(p)$ determines a coorientation of the unstable manifold $W^u(p)$, for every p.

If ind(p) − ind(q) = 1 then $W^u(p)$ and $W^s(q)$ intersect transversally along finitely many connecting orbits $\gamma(t)$ and the structure near each of the connecting orbits looks as shown on Figure 3. Note that the normal

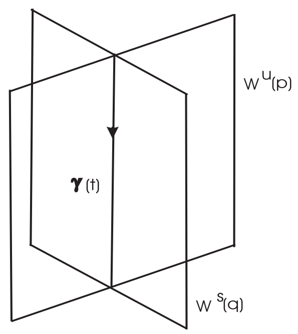

FIGURE 3. The stable and unstable manifolds along $\gamma(t)$

bundle to $W^u(p)$ along γ coincides with the normal bundle to γ in $W^s(q)$. Hence, the coorientation of $W^u(p)$ together with the natural orientation of the curve $\gamma(t)$ determine an orientation of $W^s(q)$ along γ. We set $\epsilon(\gamma) = 1$ iff this orientation coincides with the prescribed orientation of $W^s(q)$; otherwise we set $\epsilon(\gamma) = -1$.

To compare $\epsilon(\gamma)$ with $\epsilon(\tau \circ \gamma)$ we first observe that the involution τ preserves the stable and unstable manifolds $W^s(p)$ and $W^u(p)$ and for every critical point p the degrees of the restriction of τ on these submanifolds equal

$$\deg(\tau|_{W^u(p)}) = (-1)^{\text{ind(p)}}, \qquad \deg(\tau|_{W^s(p)}) = (-1)^{n-\text{ind(p)}}, \tag{19}$$

as follows from (14). Hence, applying the involution τ to the picture shown on Figure 3, we have to multiply the coorientation of $W^u(p)$ by $(-1)^{n-i-1}$ and multiply the orientation of $W^s(q)$ by $(-1)^{n-i}$. As the result the total sign will be multiplied by $(-1)^{n-i-1} \cdot (-1)^{n-i} = -1$. This proves (18) and completes the proof of the first statement of the theorem. The second statement of the Theorem follows from the first one combined with (19). □

Theorem 5. *Let M be a smooth compact connected manifold with boundary. Suppose that M is equipped with a Morse function $f : M \to [0,1]$ and with a smooth involution $\tau : M \to M$ satisfying the properties of Theorem 4. Assume that for any critical point $p \in M$ of the function f we are given a smooth closed connected submanifold*

$$X_p \subset M$$

with the following properties:

(1) X_p is τ-invariant, i.e. $\tau(X_p) = X_p$;
(2) $p \in X_p$ and for any $x \in X_p - \{p\}$, one has $f(x) < f(p)$;

(3) the function $f|_{X_p}$ is Morse and the critical points of the restriction $f|_{X_p}$ coincide with the fixed points of τ lying in X_p. In particular, $\dim X_p = \mathrm{ind(p)}$.

(4) For any fixed point $q \in X_p$ of τ the Morse indexes of f and of $f|_{X_p}$ at q coincide.

Then each submanifold X_p is orientable and the set of homology classes realized by $\{X_p\}_{p\in \mathrm{Fix}(\tau)}$ forms a free basis of the integral homology group $H_*(M;\mathbf{Z})$. In other words, we claim that the inclusion induces an isomorphism

$$\bigoplus_{\mathrm{ind(p)=i}} H_i(X_p;\mathbf{Z}) \to H_i(M;\mathbf{Z}) \tag{20}$$

for any i.

Proof of Theorem 5. First we note that each submanifold X_p is orientable. Indeed, Theorem 4 applied to the restriction $f|_{X_p}$ implies that $f|_{X_p}$ has a unique maximum and unique minimum and the top homology group $H_i(X_p;\mathbf{Z}) = \mathbf{Z}$ is infinite cyclic where $i = \dim X_p = \mathrm{ind(p)}$.

For a regular value $a \in \mathbf{R}$ of f we denote by $M^a \subset M$ the preimage $f^{-1}(-\infty, a]$. It is a compact manifold with boundary. It follows from Theorem 4 that f has a unique local minimum and therefore M^a is either empty or connected. For a slightly above the minimum value $f(p_0) = \min f(M)$ the manifold M^a is a disc and the homology of M^a is obviously realized by the submanifold $X_{p_0} \subset M^a$.

We proceed by induction on a. Our inductive statement is that the homology of M^a is freely generated by the homology classes of the submanifolds X_p where p runs over all critical points of f satisfying $f(p) \le a$.

Suppose that the statement is true for a and the interval $[a,b]$ contains a single critical value c. Let $p_1,\dots,p_r$ be the critical points of f lying in $f^{-1}(c)$. Denote

$$X = \coprod_{i=1}^{r} X_{p_i}$$

(the disjoint union). Then f induces a Morse function $\bar f : X \to \mathbf{R}$ and we set

$$X^a = \bar f^{-1}(-\infty, a].$$

Consider the Morse - Smale complexes $C_*(M^a)$, $C_*(M^b)$, $C_*(X)$ and $C_*(X^a)$; the first two are constructed using the function f and the latter two are constructed using the function $\bar f$. We have the following Mayer-Vietoris-type short exact sequence of chain complexes

$$0 \to C_*(X^a) \to C_*(X) \oplus C_*(M^a) \xrightarrow{\Phi} C_*(M^b) \to 0 \tag{21}$$

which (by the arguments indicated in the proof of Theorem 4) have trivial differentials and hence the sequence

$$0 \to H_i(X^a) \to H_i(X) \oplus H_i(M^a) \xrightarrow{\Phi} H_i(M^b) \to 0 \tag{22}$$

is exact (all homology groups have coefficients $\mathbf{Z}$). It follows from Lemma 6 below and the construction of the Morse - Smale complex (compare [11], §7) that the homomorphism Φ (which appears in (21) and (22)) coincides with the sum of the chain maps induced by the inclusions $X \to M^b$ and $M^a \to M^b$.

For $i < \dim X$ we have $H_i(X^a) \to H_i(X)$ is an isomorphism (by Theorem 4) and hence (22) implies that $H_i(M^a) \to H_i(M^b)$ is an isomorphism. For $i \geq \dim X$ we have $H_i(X^a) = 0$ and therefore $\Phi : H_i(X) \oplus H_i(M^a) \to H_i(M^b)$ is an isomorphism. This completes the step of induction. ☐

Here is a minor variation of the Morse lemma which has been used in the proof.

Lemma 6. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function having $0 \in \mathbf{R}^n$ as a nondegenerate critical point and suppose that for some $k \leq n$ the restriction $f|_{\mathbf{R}^k \times \{0\}} : \mathbf{R}^k \times \{0\} \to \mathbf{R}$ also has a nondegenerate critical point at $0 \in \mathbf{R}^k$. Then there exists a neighborhood $U \subset \mathbf{R}^n$ of 0 and a local coordinate system $x : U \to \mathbf{R}^n$ such that $x(\mathbf{R}^k \times \{0\} \cap U) \subset \mathbf{R}^k \times \{0\}$ and*

$$f(x_1, \dots, x_n) = \pm x_1^2 + \dots + \pm x_n^2 + f(0). \tag{23}$$

Proof. One simply checks that the coordinate changes in the standard proof of the Morse lemma (compare [10], §2) can be chosen so that the subspace $\mathbf{R}^k \times \{0\}$ is mapped to itself. ☐

4. The robot arm distance map

A robot arm is a simple mechanism consisting of n bars (links) of fixed length $(l_1, \dots, l_n)$ connected by revolving joints, see Figure 4. The initial point of the robot arm is fixed on the plane. The moduli space of a robot

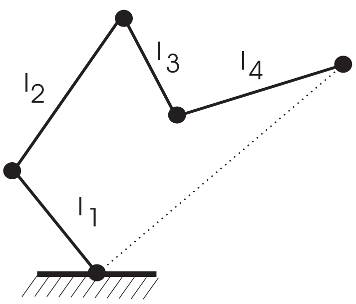

Figure 4. Robot arm.

arm (i.e. the space of its possible shapes) is

$$W = \{(u_1, \dots, u_n) \in S^1 \times \dots \times S^1\}/\mathrm{SO}(2). \tag{24}$$

Clearly, W is diffeomorphic to a torus T^{n-1} of dimension $n-1$. A diffeomorphism can be specified, for example, by assigning to a configuration $(u_1, \dots, u_n)$ the point $(1, u_2u_1^{-1}, u_3u_1^{-1}, \dots, u_{n-1}u_1^{-1}) \in T^{n-1}$ (measuring angles between the directions of the first and the other links).

Consider the moduli space of polygons M_ℓ (where $\ell = (l_1, \dots, l_n)$) which is naturally embedded into W.

We define a function on W as follows:

$$f_\ell : W \to \mathbf{R}, \qquad f_\ell(u_1, \dots, u_n) = -\left|\sum_{i=1}^{n} l_i u_i\right|^2. \tag{25}$$

Geometrically the value of f_ℓ equals the negative of the squared distance between the initial point of the robot arm to the end of the arm shown by the dotted line on Figure 4. Note that the maximum of f_ℓ is achieved on the moduli space of planar linkages $M_\ell \subset W$.

An important role play the collinear configurations, i.e. such that $u_i = \pm u_j$ for all i, j, see Figure 5. We will label such configurations by long and median subsets $J \subset \{1, \dots, n\}$ assigning to any such subset J the configuration $p_J \in W$ given by $p_J = (u_1, \dots, u_n)$ where $u_i = 1$ for $i \in J$ and $u_i = -1$ for $i \notin J$. Note that p_J lies in $M_\ell \subset W$ if and only if the subset J is median.

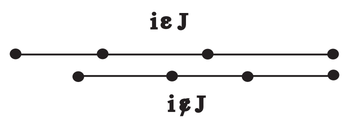

Figure 5. A collinear configuration p_J of the robot arm.

Lemma 7. *The critical points of $f_\ell : W \to \mathbf{R}$ lying in $W - M_\ell$ are exactly the collinear configurations p_J corresponding to long subsets $J \subset \{1, 2, \dots, n\}$. Each p_J, viewed as a critical point of f_ℓ, is nondegenerate in the sense of Morse and its Morse index equals $n - |J|$.*

This lemma is well-known. It can be found as Proposition 3.3 in [14] and as combination of Theorems 3.1 and 3.2 in [1]; in both these references slightly different notations were used.

5. Proof of Theorem 1.

Consider the moduli space W of the robot arm (defined by (24)) with the function $f_\ell : W \to \mathbf{R}$ (defined by (25)). There is an involution

$$\tau : W \to W \tag{26}$$

given by

$$\tau(u_1, \dots, u_n) = (\bar{u}_1, \dots, \bar{u}_n). \tag{27}$$

Here the bar denotes complex conjugation, i.e. the reflection with respect to the real axis. It is obvious that formula (27) maps SO(2)-orbits into SO(2)-orbits and hence defines an involution on W. The fixed points of τ are the collinear configurations of the robot arm, i.e. the critical points of f_ℓ in $W - M_\ell$, see Lemma 7. Our plan it to apply Theorems 4 and 5 to the sublevel sets

$$W^a = f_\ell^{-1}(-\infty, a] \tag{28}$$

of f_ℓ. Recall that the values of f_ℓ are nonpositive and the maximum is achieved on the submanifold $M_\ell \subset W$. From Lemma 7 we know that the critical points of f_ℓ are the collinear configurations p_J. The latter are labelled by long subsets $J \subset \{1, \dots, n\}$ and $p_J = (u_1, \dots, u_n)$ where $u_i = 1$ for $i \in J$ and $u_i = -1$ for $i \notin J$. One has

$$f_\ell(p_J) = -(L_J)^2. \tag{29}$$

Here $L_J = \sum_{i=1}^n l_i u_i$ with $p_J = (u_1, \dots, u_n)$.

The number a which appears in (28) will be chosen so that

$$-(L_J)^2 < a < 0 \tag{30}$$

for any long subset J such that the manifold W^a contains all the critical points p_J. The situation is shown schematically on Figure 6.

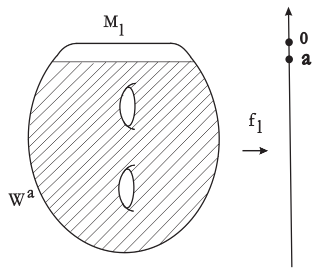

FIGURE 6. Function $f_\ell : W \to \mathbf{R}$ and the manifold W^a.

For each subset $J \subset \{1, \dots, n\}$ we denote by ℓ_J the length vector obtained from $\ell = (l_1, \dots, l_n)$ by integrating all links l_i with $i \in J$ into one link. For example, if $J = \{1, 2\}$ then $\ell_J = (l_1 + l_2, l_3, \dots, l_n)$. We denote by W_J the moduli space of the robot arm with the length vector ℓ_J. It is obvious that W_J is diffeomorphic to a torus $T^{n-|J|}$. We view W_J as being naturally embedded into W. Note that the submanifold $W_J \subset W$ is disjoint from M_ℓ (in other words, W_J contains no closed configurations) if and only if the subset $J \subset \{1, \dots, n\}$ is long.

Lemma 8. *Let $J \subset \{1, \dots, \}$ be a long subset. The submanifold $W_J \subset W$ has the following properties:*

(1) W_J is invariant with respect to the involution $\tau : W \to W$;

(2) *the restriction of f_ℓ onto W_J is a Morse function having as its critical points the collinear configurations p_I where I runs over all subsets $I \subset \{1, \dots, n\}$ containing J.*
(3) *for any such p_I the Morse indexes of f_ℓ and of $f_\ell|_{W_J}$ at p_I coincide.*
(4) *in particular, $f|_{W_J}$ achieves its maximum at $p_J \in W_J$.*

Proof. (1) is obvious. Statements (2) and (3) follow from Lemma 7 applied to the restriction of f_ℓ onto W_J. Here we use the assumption that J is long. Under this assumption the long subset for the integrated length vector ℓ_J are in one-to-one correspondence with the long subsets $I \subset \{1, \dots, n\}$ containing J. Statement (4) follows from (3) as the Morse index of $f_\ell|_{W_J}$ at point p_J equals $n - |J| = \dim W_J$. □

Applying Theorems 4 and 5 and taking into account Lemma 8 we obtain:

Corollary 9. *One has:*
(1) *If a satisfies (30) then the manifold W^a (see (28)) contains all submanifolds W_J where $J \subset \{1, \dots, n\}$ is an arbitrary long subset.*
(2) *The homology classes of the submanifolds W_J form a free basis of the integral homology group $H_*(W^a; \mathbf{Z})$.*

Next we examine the homomorphism

$$\phi_* : H_i(W^a; \mathbf{Z}) \to H_i(W; \mathbf{Z}) \tag{31}$$

induced by the inclusion $\phi : W^a \to W$.

Below we will assume that $l_1 \geq l_j$ for all $j \in \{1, \dots, n\}$, i.e. l_1 is the longest link. This may always be achieved by relabelling.

We describe a specific basis of the homology $H_*(W; \mathbf{Z})$. For any subset $J \subset \{1, 2, \dots, n\}$ we denote by W_J the moduli space of configurations of the robot arm with length vector ℓ_J where all links l_i with $i \in J$ are integrated into a single link. Note that W_J is naturally embedded into W and

$$W_J \cap M_\ell = \emptyset$$

if and only if the set J is long. Since W is homeomorphic to the torus T^{n-1}, it is easy to see that a basis of the homology group $H_*(W; \mathbf{Z})$ is formed by the homology classes of the submanifolds W_J where $J \subset \{1, \dots, n\}$ runs over all subsets containing 1. We will denote the homology class of W_J by

$$[W_J] \in H_{n-|J|}(W; \mathbf{Z}). \tag{32}$$

Assuming that $J, J' \subset \{1, \dots, n\}$ are two subsets with $|J| + |J'| = n + 1$ the classes $[W_J]$ and $[W_{J'}]$ have complementary dimensions in W and their intersection number is given by

$$[W_J] \cdot [W_{J'}] = \begin{cases} \pm 1, & \text{if} \quad |J \cap J'| = 1, \\ \\ 0, & \text{if} \quad |J \cap J'| > 1. \end{cases} \tag{33}$$

Indeed, if $J \cap J' = \{i_0\}$ then $W_J \cap W_{J'}$ consists of a single point $\{p\}$, the moduli space of a robot arm with all links integrated into one link. Let us show that the intersection $W_J \cap W_{J'}$ is transversal. A tangent vector to W at $p = (u_1, \dots, u_n)$ can be labelled by a vector $w = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$ (an element of the Lie algebra of the torus T^n) viewed up to adding vectors of the form $(\lambda, \lambda, \dots, \lambda)$. Such a tangent vector w is tangent to the submanifold W_J iff $\lambda_i = \lambda_j$ for all $i, j \in J$. Given w as above it can be written as

$$w = w' + w'' + (\lambda_{i_0}, \dots, \lambda_{i_0})$$

where w' has coordinates 0 on places $i \in J$ and coordinates $\lambda_i - \lambda_{i_0}$ on places $i \notin J$; coordinates of w'' vanish on places $i \notin J$ and are $\lambda_i - \lambda_{i_0}$ on places $i \in J$. Hence every tangent vector to W is a sum of a tangent vector to W_J and a tangent vector to $W_{J'}$.

Now suppose that $|J \cap J'| > 1$. We will show that then the submanifold $W_{J'}$ can be continuously deformed inside W to a submanifold $W'_{J'}$ such that $W_J \cap W'_{J'} = \emptyset$. This would prove the second claim in (33). Let us assume that $\{1, 2\} \subset J \cap J'$. Define $g_t : W_{J'} \to W$ by

$$g_t(u_1, \dots, u_n) = (e^{i\theta t}u_1, u_2, \dots, u_n), \quad t \in [0, 1].$$

Here θ satisfies $0 < \theta < \pi$. Then $W'_{J'} = g_1(W_{J'})$ is clearly disjoint from W_J; indeed, the links l_1 and l_2 are parallel in W_J and make an angle θ in $W'_{J'}$.

It follows that the intersection form in the basis $[W_J], [W_{J'}] \in H_*(W; \mathbf{Z})$, where $J \ni 1, J' \ni 1$, has a very simple form:

$$[W_J] \cdot [W_{J'}] = \begin{cases} \pm 1, & \text{if} \quad J \cap J' = \{1\}, \\ \\ 0, & \text{if} \quad |J \cap J'| > 1. \end{cases} \tag{34}$$

In particular, given $[W_J]$ with $1 \in J$, its "dual" homology class $\in H_*(W; \mathbf{Z})$ (in the sense of the homological intersection form) equals $[W_K]$ where $K = CJ \cup \{1\}$; here CJ denotes the complement of J in $\{1, \dots, n\}$.

Denote by $A_* \subset H_*(W^a; \mathbf{Z})$ (correspondingly, $B_* \subset H_*(W^a; \mathbf{Z})$) the subgroup generated by the homology classes $[W_J]$ where $J \subset \{1, \dots, n\}$ is long and contains 1 (correspondingly, J is long and $1 \notin J$). Then

$$H_i(W^a; \mathbf{Z}) = A_i \oplus B_i. \tag{35}$$

Similarly, one has

$$H_i(W; \mathbf{Z}) = A_i \oplus C_i \oplus D_i, \tag{36}$$

where:

- A_* is as above;
- $C_* \subset H_*(W; \mathbf{Z})$ is the subgroup generated by the homology classes $[W_J]$ with $J \subset \{1, \dots, n\}$ short and $1 \in J$;
- D_* is the subgroup generated by the classes $[W_J] \in H_*(W; \mathbf{Z})$ where J is median and contains 1.

It is clear that ϕ_* (see (31)) is identical when restricted to A_i, compare (35) and (36). We claim that the image $\phi_*(B_i)$ is contained in A_i. This would follow once we show that

$$[W_J] \cdot [W_K] = 0 \tag{37}$$

assuming that $[W_J] \in B_i$ and $[W_K]$ is the dual of a class $[W_{J'}] \in C_i$ or $[W_{J'}] \in D_i$, see (34). We have

(1) J is long and $1 \notin J$,
(2) J' is short or median and $1 \in J'$,
(3) $|J| = |J'|$,
(4) $K = CJ' \cup \{1\}$.

Here CJ' denotes the complement of J' in $\{1, \dots, n\}$. By (33), to prove (37) we have to show that under the above conditions one has $|J \cap K| > 1$. Indeed, suppose that $|J \cap K| = 1$, i.e. $J \cap K = \{j\}$, a single element subset. Then J' is obtained from J by removing the index j and adding the index 1 which leads to a contradiction: indeed, J is long, $l_j \le l_1$ and J' is either short or median.

Corollary 10. *The kernel of the homomorphism*

$$\phi_i : H_i(W^a; \mathbf{Z}) \to H_i(W; \mathbf{Z})$$

has rank equal[1] *to* $\operatorname{rk} B_i$ *and the cokernel has rank* $\operatorname{rk} C_i + \operatorname{rk} D_i$.

Below we skip the coefficient group $\mathbf{Z}$ from the notations.

One has

$$H_j(W, W^a) \simeq H_j(N, \partial N) \simeq H^{n-1-j}(N) \simeq H^{n-1-j}(M_\ell). \tag{38}$$

Here N denotes the preimage $f_\ell^{-1}([a, 0])$. Note that M_ℓ is a deformation retract of N. Indeed, consider $M_\ell \subset N'' \subset N' \subset N$ where N' is a regular neighborhood of M_ℓ in N and N'' is a sublevel set $N'' = f_\ell^{-1}([a', 0])$ and $a < a' < 0$ is such that N'' is contained in N'. Since $M_\ell \subset N'$ and $N'' \subset N$ are deformation retracts, we have the following diagram

$$\begin{array}{ccc} M_\ell & \stackrel{r'}{\leftarrow} & N' \\ i \downarrow & \nearrow j & \downarrow k \\ N'' & \stackrel{r}{\leftarrow} & N \end{array}$$

where i, j, k are inclusions and $r'ji = 1_{M_\ell}$, $jir' \simeq 1_{N'}$, $rkj = 1_{N''}$, $kjr \simeq 1_N$. It follows that $g = r'jr : N \to M_\ell$ is a deformation retraction.

Hence we obtain the following short exact sequence

$$0 \to \operatorname{coker}(\phi_{n-1-j}) \to H^j(M_\ell) \to \ker \phi_{n-2-j} \to 0 \tag{39}$$

which splits since the kernel of ϕ_{n-2-j} is isomorphic to B_{n-2-j} (see above) and hence it is free abelian.

This proves that the cohomology $H^*(M_\ell)$ has no torsion and therefore the homology $H_*(M_\ell)$ is free as well (by the Universal Coefficient Theorem).

[1]Note that the kernel of ϕ_i (viewed as a subgroup) is distinct from B_i in general.

The cokernel of ϕ_{n-1-j} is isomorphic to $C_{n-1-j} \oplus D_{n-1-j}$ as we established earlier. We find that the rank of cokerϕ_{n-1-j} equals the number of subsets $J \subset \{1, \dots, n\}$ which are short or median and have cardinality $|J| = j + 1$. In other words,

$$\text{rk}(\text{coker}\phi_{n-1-j}) = a_j + b_j, \tag{40}$$

where we use the notation introduced in the statement of Theorem 1.

The rank of the kernel of ϕ_{n-2-j} equals the rank of B_{n-2-j}, i.e. the number of long subsets $J \subset \{2, \dots, n\}$ of cardinality $|J| = j + 2$. Passing to the complements, we find

$$\text{rk}(\ker \phi_{n-2-j}) = a_{n-3-j} \tag{41}$$

i.e. the number of short subsets containing 1 with $|J| = n - 2 - j$.

Combining (40), (41) with the exact sequence (39) we finally obtain

$$\text{rk}\, H_j(M_\ell) = \text{rk}\, H^j(M_\ell) = a_j + b_j + a_{n-3-j}.$$

This completes the proof, compare (1).

6. Proofs of Theorems 2 and 3

The proofs are based on Theorem 1 and are purely combinatorial. Let $\ell = (l_1, \dots, l_n)$ be a length vector. Without loss of generality we may assume that $l_1 \le l_2 \le \cdots \le l_n$. By Theorem 1 the sum of Betti numbers of M_ℓ equals twice the number of short subsets plus the number of median subsets of $\{1, \dots, n\}$, containing n. We show that the number of such subsets is bounded above by B_n (given by (11)); moreover, we show that it is bounded above by $B'_n = 2 \cdot B_{n-1}$ (see (12)) assuming additionally that ℓ is generic and n is even.

We will treat simultaneously both cases n even and n odd. Denote $r = [(n-1)/2]$ so that $n = 2r + 2$ for n even and $n = 2r + 1$ for n odd.

For $1 \le i \le n$ we denote by $S_i(\ell)$ (respectively, $M_i(\ell)$) the number of short (respectively, median) subsets of $\{1, \dots, n\}$ containing the subset $\{n - i + 1, n - i + 2, \dots, n\}$. Clearly, $S_{r+1}(\ell) = M_{r+1}(\ell) = 0$ for n odd and $S_{r+1}(\ell) = 0$, $M_{r+1}(\ell) \le 1$ for n even.

We claim that

$$2 \cdot S_i(\ell) + M_i(\ell) \,\le\, 2^{n-i} - \sum_{j=r-i+1}^{r} \binom{n-i}{j} \tag{42}$$

for all $1 \le i \le r + 1$. For $i = 1$ inequality (42) gives

$$2 \cdot S_1(\ell) + M_1(\ell) \,\le\, 2^{n-1} - \binom{n-1}{r} \,=\, B_n$$

which is equivalent to our goal (11). We will prove (42) by induction on n and by descending induction on i.

For n odd and $i = r + 1$ inequality (42) gives $2 \cdot S_{r+1}(\ell) + M_{r+1}(\ell) \le 0$, which follows from our remark above. Similarly, for n even and $i = r + 1$

inequality (42) states $2 \cdot S_{r+1}(\ell) + M_{r+1}(\ell) \le 1$, which is obviously true, see above. These two remarks serve as the initial step of induction.

Assume now that inequality (42) is true (a) for $i+1$ and (b) for all i and all length vectors $\ell' = (l'_1, \dots, l'_m)$ with $m < n$.

One can write

$$S_i(\ell) = S_{i+1}(\ell) + S'_i(\ell), \quad M_i(\ell) = M_{i+1}(\ell) + M'_i(\ell) \tag{43}$$

where $S'_i(\ell)$ and $M'_i(\ell)$ denote the numbers of short and median subsets of $\{1, \dots, n\}$ containing $\{n-i+1, \dots, n\}$ and not containing $n-i$. One observes that

$$S'_i(\ell) \le S_{i-1}(\tilde{\ell}), \quad \text{and} \quad S'_i(\ell) + M'_i(\ell) \le S_{i-1}(\tilde{\ell}) + M_{i-1}(\tilde{\ell}), \tag{44}$$

where

$$\tilde{\ell} = (l_1, l_2, \dots, l_{n-i-1}, l_{n-i+2}, \dots, l_n). \tag{45}$$

Hence, using (43) and (44), we obtain

$$2S_i(\ell) + M_i(\ell) \le [2S_{i+1}(\ell) + M_{i+1}(\ell)] + \left[2S_{i-1}(\tilde{\ell}) + M_{i-1}(\tilde{\ell})\right]. \tag{46}$$

By our inductive hypothesis,

$$\begin{aligned} 2 \cdot S_{i+1}(\ell) + M_{i+1}(\ell) &\le 2^{n-i-1} - \sum_{j=r-i}^{r} \binom{n-i-1}{j} \\ &= 2^{n-i-1} - \sum_{j=r-i+1}^{r-1} \binom{n-i-1}{j-1} + \binom{n-i}{r} \end{aligned}$$

and

$$2S_{i-1}(\tilde{\ell}) + M_{i-1}(\tilde{\ell}) \le 2^{n-i-1} - \sum_{j=r-i+1}^{r-1} \binom{n-i-1}{j}$$

Adding the last two inequalities and taking into account (46) we obtain

$$\begin{aligned} 2S_i(\ell) + M_i(\ell) \quad &\le \quad 2^{n-i} - \sum_{j=r-i+1}^{r-1} \left[\binom{n-i-1}{j} + \binom{n-i-1}{j-1}\right] \\ &+ \quad \binom{n-i}{r} = 2^{n-i} - \sum_{j=r-i+1}^{r} \binom{n-i}{j}. \end{aligned}$$

This completes the proof of Theorem 2.

To prove Theorem 3 we assume that n is even, $n = 2r+2$, and $\ell = (l_1, \dots, l_n)$ is a generic length vector where $l_1 \le l_2 \le \dots \le l_n$. We replace the inductive hypothesis (42) by

$$2 \cdot S_i(\ell) \le 2^{n-i} - 2 \cdot \sum_{j=r-i+1}^{r} \binom{n-i-1}{j} \tag{47}$$

for $1 \le i \le r+1$. For $i = 1$ inequality (47) gives the desired inequality

$$2 \cdot S_1(\ell) \le 2^{n-1} - 2 \cdot \binom{2r}{r} = B'_n,$$

compare (12). For $i = r+1$ inequality (47) states $S_{r+1}(\ell) \le 0$ which is obviously correct; this statement will be the base of induction. To perform the step of induction we use inequalities (43) and (44) which are valid in the case of n even as well. We find

$$2 \cdot S_{i+1}(\ell) \le 2^{n-i-1} - 2 \cdot \sum_{j=r-i}^{r} \binom{n-i-2}{j}$$

and

$$2 \cdot S'_i(\ell) \le 2^{n-i-1} - 2 \cdot \sum_{j=r-i+1}^{r-1} \binom{n-i-2}{j}$$

(both by the induction hypothesis) and adding the last two inequalities, using (43), and performing transformations similar to the odd case, we obtain (47).

This completes the proof of Theorem 3.

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