# ON THE WHITEHEAD GROUP OF NOVIKOV RINGS ASSOCIATED TO IRRATIONAL HOMOMORPHISMS

DIRK SCHÜTZ

Abstract. Given a homomorphism $\xi : G \to \mathbb{R}$ we show that the natural map $i_* : \mathrm{Wh}(G) \to \mathrm{Wh}(G;\xi)$ from the Whitehead group of $G$ to the Whitehead group of the Novikov ring is surjective. The group $\mathrm{Wh}(G;\xi)$ is of interest for the simple chain homotopy type of the Novikov complex. It also contains the Latour obstruction for the existence of a nonsingular closed 1-form within a fixed cohomology class $\xi \in H^1(M;\mathbb{R})$, where $M$ is a closed connected smooth manifold.

## 1. Introduction

Given a group $G$ and a homomorphism $\xi : G \to \mathbb{R}$ to the additive group of real numbers the Novikov ring $\widehat{\mathbb{Z}G}_\xi$ is a completion of the ordinary group ring $\mathbb{Z}G$. Elements of $\widehat{\mathbb{Z}G}_\xi$ can be thought of as functions $\lambda : G \to \mathbb{Z}$ such that for every real number $r \in \mathbb{R}$ there are only finitely many $g \in G$ with $\lambda(g) \neq 0$ and $\xi(g) \geq r$.

This ring arises naturally in the Morse theory of closed 1-forms on closed smooth manifolds $M$ and was introduced by Novikov [14] for injective $\xi$ and more generally by Sikorav [29]. A closed 1-form $\omega$ on $M$ induces a homomorphism $\xi = \xi_{[\omega]} : \pi_1(M) \to \mathbb{R}$ via its cohomology class. Provided that $\omega$ satisfies a Morse condition one can define the so called Novikov complex $C_*(M,\omega)$. This is a chain complex which is finitely generated free over $\widehat{\mathbb{Z}G}_\xi$, where $G$ is a quotient of $\pi_1(M)$ by a normal subgroup contained in $\ker \xi$. For details on several constructions we refer the reader to Novikov [15], Latour [12], Pajitnov [17], Farber [6] or Schütz [25]. It turns out that its chain homotopy type is that of $C_*(M;\widehat{\mathbb{Z}G}_\xi)$.

In recent years there has been considerable interest also in the simple homotopy type of the Novikov complex, see Latour [12], Pajitnov [18], Damian [4], Schütz [24] or Cornea and Ranicki [3]. Notably Latour [12] introduced the Whitehead group of the Novikov ring $\mathrm{Wh}(G;\xi)$, a quotient of $K_1(\widehat{\mathbb{Z}G}_\xi)$ by so called trivial units. These trivial units consist of $\pm g \in \widehat{\mathbb{Z}G}_\xi$ for all $g \in G$ and units of the form $1 - a \in \widehat{\mathbb{Z}G}_\xi$ where $a : G \to \mathbb{Z}$ satisfies $a(g) = 0$ for $\xi(g) \geq 0$.

An important feature of this group is that it contains an obstruction for the existence of a nonsingular closed 1-form $\omega$ in a fixed cohomology class. More precisely, Latour [12] gives two conditions for a nonzero cohomology class $\xi \in H^1(M;\mathbb{R})$. The first, homotopy theoretical condition, assures that the Novikov homology vanishes. The second condition is then that the Whitehead torsion of the Novikov complex, measured in $\mathrm{Wh}(G;\xi)$, vanishes. We give a brief account of this in Section 7.

---

2000 *Mathematics Subject Classification.* Primary 19B28; Secondary 57Q10.

*Key words and phrases.* Whitehead group, Novikov ring, Latour obstruction.

For this reason we would like to get a better understanding of $\mathrm{Wh}(G;\xi)$. There is an obvious homomorphism $i_*:\mathrm{Wh}(G)\to\mathrm{Wh}(G;\xi)$ from the ordinary Whitehead group of $G$ induced by the inclusion $\mathbb{Z}G\subset\widehat{\mathbb{Z}G}_\xi$. Although it is known that $\mathrm{Wh}(G)$ can be very complicated, there are also many examples where this group vanishes. The main theorem of this paper states that $i_*$ is surjective, so that the vanishing of $\mathrm{Wh}(G)$ indeed implies the vanishing of $\mathrm{Wh}(G;\xi)$.

**Theorem 1.1.** *Let $G$ be a group and $\xi:G\to\mathbb{R}$ a homomorphism. Then $i_*:\mathrm{Wh}(G)\to\mathrm{Wh}(G;\xi)$ is surjective.*

In the case where $\xi$ factors through the integers this theorem was known before. Namely it follows immediately from the Main Theorem in Pajitnov and Ranicki [20]. In the case where $G=H\times\mathbb{Z}$ and $\xi$ is projection to $\mathbb{Z}$ it also follows from Pajitnov [18, Prop.7.7].

In [20] actually more is shown. If $\xi$ is a homomorphism to the integers, then the Novikov ring can be identified with a twisted Laurent series ring $A_\rho((t))$. Now Pajitnov and Ranicki obtain a direct sum decomposition for $K_1(A_\rho((t)))$ analogous to the Bass-Heller-Swan decomposition of $K_1(A[t,t^{-1}])$. From this decomposition, which we describe in Section 7, it follows that $i_*$ is not an isomorphism in general.

Yet $\mathrm{Wh}(G;\xi)$ cannot be significantly less complicated than $\mathrm{Wh}(G)$, as the next theorem shows.

**Theorem 1.2.** *Let $G$ be a group and $\xi:G\to\mathbb{R}$ a homomorphism. Then the diagonal map $\mathrm{Wh}(G)\to\mathrm{Wh}(G;\xi)\oplus\mathrm{Wh}(G;-\xi)$ is injective.*

If $\xi$ factors through the integers, this follows immediately from the decomposition of Pajitnov and Ranicki [20], and the methods used to prove Theorem 1.1 allow us to deduce the general case from that.

In order to prove Theorem 1.1 it is not important that the Novikov ring is formed over the integers. Also there is no need to factor out trivial units of the form $\pm g$ for $g\in G$ as they are already in the group ring. Let $\overline{W}_\xi$ be the subgroup of $K_1(\widehat{RG}_\xi)$ generated by units of the form $1-a\in\widehat{RG}_\xi$ with $a(g)=0$ for $\xi(g)\geq 0$. The more general version then reads

**Theorem 1.3.** *Let $G$ be a group, $\xi:G\to\mathbb{R}$ a homomorphism and $R$ a ring with unit. Then $i_*:K_1(RG)\to K_1(\widehat{RG}_\xi)/\overline{W}_\xi$ is surjective.*

In order to prove Theorem 1.3 we want to apply the methods of Pajitnov and Ranicki [20]. This does not work directly since their techniques make strong use of the Laurent series ring description. But in general the Novikov ring cannot be described as a twisted Laurent series ring in several variables. Instead we will approximate the Novikov ring by subrings to which the techniques of [20] can be applied inductively.

We start by looking at finitely generated groups $G$. Then $G/\ker\xi\cong\mathbb{Z}^k$ for some $k\geq 1$. The first step is to show that every $\tau_0\in K_1(\widehat{RG}_\xi)/\overline{W}_\xi$ can be represented by a matrix $A$ invertible over a subring $\Lambda_0$ depending on $\tau_0$. This ring has the property that there exist surjective homomorphisms $\xi_i:G\to\mathbb{Z}$ for $i=1,\ldots,k$ such that $\Lambda_0$ is also a subring of every $\widehat{RG}_{\xi_i}$. Now $RG_{\xi_i}$ can be identified as a twisted Laurent series ring and in particular has a twisted power series subring denoted $\widehat{RG}^{o}_{\xi_i}$. We

then get a sequence of subrings $\Lambda_k \subset \ldots \subset \Lambda_1 \subset \Lambda_0$, where $\Lambda_j$ is also a subring of $\widehat{RG}^o_{\xi_i}$ for $i \leq j$.
The second step is then to show that given $\tau_j \in K_1(\Lambda_j)$, we can find $\tau_G \in K_1(RG)$ and $\tau_{j+1} \in K_1(\Lambda_{j+1})$ such that $i_*\tau_j = i_*\tau_G + i_*\tau_{j+1} \in K_1(\widehat{RG}_\xi)$. This implies the theorem since $i_*\tau_k \in \overline{W}_\xi$.
The case of a group which is not finitely generated is deduced by a direct limit argument.

## 2. Novikov rings

Let $G$ be a group, $\xi : G \to \mathbb{R}$ a homomorphism to the additive group of real numbers and $R$ a ring with unit. We denote by $R^G$ the abelian group of all functions $\lambda : G \to R$. For $\lambda \in R^G$ denote $\operatorname{supp}\lambda = \{g \in G \mid \lambda(g) \neq 0\}$.

**Definition 2.1.** The *Novikov ring* $\widehat{RG}_\xi$ is defined as

$$\widehat{RG}_\xi \quad = \quad \{\lambda \in R^G \mid \forall\, r \in \mathbb{R} \quad \operatorname{supp}\lambda \cap \xi^{-1}([r,\infty)) \text{ is finite}\}$$

with $\lambda \cdot \mu(g) = \sum \lambda(g_1)\mu(g_2)$ for $\lambda, \mu \in \widehat{RG}_\xi$. The sum is taken over all $g_1, g_2 \in G$ with $g_1 g_2 = g$.

For $\lambda \in \widehat{RG}_\xi$ let

$$\|\lambda\|_\xi \quad = \quad \inf\{t \in (0,\infty) \mid \operatorname{supp}\lambda \subset \xi^{-1}((-\infty, \log t])\}$$

be the *norm* of $\lambda$ with respect to $\xi$. Note that $\widehat{RG}_\xi$ is a completion of the group ring $RG$ with respect to the metric induced by this norm. We can extend the definition of the norm to $n \times m$ matrices over $\widehat{RG}_\xi$ by setting

$$\|A\|_\xi \quad = \quad \max\{\|A_{ij}\|_\xi \mid i \in \{1,\ldots,n\}, j \in \{1,\ldots,m\}\}\,.$$

It is easy to see that

$$\|A \cdot B\|_\xi \quad \leq \quad \|A\|_\xi \cdot \|B\|_\xi \tag{1}$$

for an $n \times m$ matrix $A$ and an $m \times k$ matrix $B$.
Since the multiplication in $\widehat{RG}_\xi$ does not depend on $\xi$ and $\widehat{RG}_\xi$ is a subgroup of $R^G$, we can intersect Novikov rings for different homomorphisms $\xi : G \to \mathbb{R}$ and obtain a ring again.
Define

$$\widehat{RG}^o_\xi \quad = \quad \{\lambda \in \widehat{RG}_\xi \mid \|\lambda\|_\xi \leq 1\}.$$

Because of (1) we get that $\widehat{RG}^o_\xi$ is a subring of $\widehat{RG}_\xi$.

**Lemma 2.2.** *For $i = 1,\ldots,k$ let $\xi_i : G \to \mathbb{R}$ be a homomorphism and $t_i \in (0,\infty)$. Denote $\xi = \sum_{i=1}^k t_i\xi_i : G \to \mathbb{R}$. Then*

1. *$\widehat{RG}_{\xi_1} \cap \ldots \cap \widehat{RG}_{\xi_k}$ is a subring of $\widehat{RG}_\xi$.*
2. *$\widehat{RG}^o_{\xi_1} \cap \ldots \cap \widehat{RG}^o_{\xi_k}$ is a subring of $\widehat{RG}^o_\xi$.*

*Proof.* It is enough to assume $k = 2$. Since $\widehat{RG}_{\xi_1} = \widehat{RG}_{t_1\xi_1}$ for $t_1 > 0$ we can also assume $t_1 = t_2 = 1$. Let $\lambda \in \widehat{RG}_{\xi_1} \cap \widehat{RG}_{\xi_2}$. There is $r_2 \in \mathbb{R}$ with $\operatorname{supp}\lambda \cap \xi_2^{-1}([r_2,\infty)) = \emptyset$. For $r \in \mathbb{R}$ we now get

$$\operatorname{supp}\lambda \cap \xi^{-1}([r,\infty)) \subset \operatorname{supp}\lambda \cap \xi_1^{-1}([r - r_2,\infty)).$$

Since $\operatorname{supp}\lambda \cap \xi_1^{-1}([r - r_2,\infty))$ is finite, we get (1).

To see (2) note that for $\lambda$ in the intersection we get that $g \in \operatorname{supp}\lambda$ implies that $\xi_i(g) \leq 0$, hence also $\xi(g) \leq 0$. $\square$

Lemma 2.2 shows that the intersection $\widehat{RG}_{\xi_1} \cap \widehat{RG}_{\xi_2}$ is not just a subring of each Novikov ring, but also a subring of the Novikov ring corresponding to a convex combination of $\xi_1$ and $\xi_2$.

## 3. Torsion

Let $R$ be a ring with unit. Then $K_1(R)$ is the abelian group generated by $\tau(f)$ for each automorphism $f : M \to M$, where $M$ is a finitely generated projective left $R$-module subject to the following relations.

(1) For a short exact sequence of automorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

we have $\tau(e) - \tau(f) + \tau(g) = 0$.

(2) For automorphisms $f, g : M \to M$ we have $\tau(f \circ g) = \tau(f) + \tau(g)$.

Notice that for every automorphism $f : M \to M$ of the finitely generated projective $R$-module $M$ there exists an automorphism $g : R^n \to R^n$ of the finitely generated free $R$-module $R^n$ with $\tau(f) = \tau(g)$. We can think of $g$ as an invertible $n \times n$ matrix over $R$. This leads to another way to describe $K_1(R)$. Let $GL(n,R)$ be the group of invertible $n \times n$ matrices over $R$. We have the standard inclusion $GL(n,R) \subset GL(n+1,R)$ and let $GL(R)$ be the direct limit. Then

$$K_1(R) \quad = \quad GL(R)/[GL(R), GL(R)],$$

the abelianization of $GL(R)$. Indeed the commutator subgroup is generated by elementary matrices, see Cohen [1, §10]. Recall that an elementary matrix over a ring $R$ with unit is an $n \times n$ matrix $E^x_{ij}$ for $i \neq j$ and $x \in R$ which has 1 in every diagonal spot, $x$ in the $(i,j)$-spot and zero everywhere else.

Let $\xi : G \to \mathbb{R}$ be a homomorphism and let $H = \ker\xi$. Restriction defines a ring homomorphism $\varepsilon : \widehat{RG}^{o}_{\xi} \to RH$ with $\varepsilon \circ i = \mathrm{id} : RH \to RH$, where $i : RH \to \widehat{RG}^{o}_{\xi}$ is the natural inclusion. Let $a \in \widehat{RG}^{o}_{\xi}$ satisfy $\|a\|_\xi < 1$. Then $1-a$ is a unit in $\widehat{RG}^{o}_{\xi}$ with inverse $1 + a + a^2 + \ldots$ and as such it represents a torsion $\tau(1-a) \in K_1(\widehat{RG}^{o}_{\xi})$. Let $W_\xi \subset K_1(\widehat{RG}^{o}_{\xi})$ be the subgroup of such torsions.

The proof of the next proposition is basically contained in Pajitnov [17, Lm.1.1], compare also Pajitnov and Ranicki [20, Prop.2.11].

**Proposition 3.1.** *We have*

$$K_1(\widehat{RG}_\xi^o) \quad = \quad K_1(RH) \oplus W_\xi$$

*Proof.* We get $K_1(\widehat{RG}_\xi^o) = K_1(RH) \oplus \ker(\varepsilon_* : K_1(\widehat{RG}_\xi^o) \to K_1(RH))$ by functoriality. Let $B$ be a matrix with $\tau(B) \in \ker \varepsilon_*$. Then there exist matrices $E, E' \in [GL(RH), GL(RH)]$ with $E\varepsilon(B)E' = I$. Note that $E, E' \in [GL(\widehat{RG}_\xi^o), GL(\widehat{RG}_\xi^o)]$, so $EBE' = I - A$ with $\|A\|_\xi < 1$. Using elementary row and column operations it follows that $\tau(I - A) = \tau(1 - a)$ for some $1 - a \in \widehat{RG}_\xi$ with $\|a\|_\xi < 1$. □

For $K_1(\widehat{RG}_\xi)$ we do not obtain a similar formula as in Proposition 3.1, instead we will content ourselves with a certain quotient of this group. Let $\overline{W}_\xi$ be the image of $W_\xi$ under the natural map $i_* : K_1(\widehat{RG}_\xi^o) \to K_1(\widehat{RG}_\xi)$. Sometimes we will write $\overline{W}_\xi(G)$ to emphasize the group $G$. The inclusion of rings $RG \subset \widehat{RG}_\xi$ induces a natural homomorphism

$$i_* : K_1(RG) \to K_1(\widehat{RG}_\xi)$$

and the composition of this with the projection to the quotient $K_1(\widehat{RG}_\xi)/\overline{W}_\xi$ will be denoted by $i_*$ as well. Our main result now reads

**Theorem 3.2.** *Let $G$ be a group, $\xi : G \to \mathbb{R}$ a homomorphism and $R$ a ring with unit. Then $i_* : K_1(RG) \to K_1(\widehat{RG}_\xi)/\overline{W}_\xi$ is surjective.*

For geometric applications the following quotients are particularly important.

**Definition 3.3.** Let $G$ be a group and $\xi : G \to \mathbb{R}$ be a homomorphism. Then we define the *Whitehead group of $G$* as

$$\mathrm{Wh}(G) \quad = \quad K_1(\mathbb{Z}G)/\langle \tau(\pm g) \mid g \in G \rangle$$

and the *Whitehead group of the Novikov ring* as

$$\mathrm{Wh}(G; \xi) \quad = \quad K_1(\widehat{\mathbb{Z}G}_\xi)/\langle \tau(\pm g), \tau(1 - a) \mid g \in G, 1 - a \in \widehat{\mathbb{Z}G}_\xi, \|a\|_\xi < 1 \rangle.$$

The Whitehead group $\mathrm{Wh}(G; \xi)$ of the Novikov ring first appeared in Latour [12].

**Corollary 3.4.** *Let $G$ be a group and $\xi : G \to \mathbb{R}$ a homomorphism. Then $i_* : \mathrm{Wh}(G) \to \mathrm{Wh}(G; \xi)$ is surjective.*

Before we proof Theorem 3.2 we will first take a closer look at homomorphisms of the form $\xi : \mathbb{Z}^n \to \mathbb{R}$.

**Remark 3.5.** In the case of an injective homomorphism $\xi : \mathbb{Z}^n \to \mathbb{R}$ it was shown by Jean-Claude Sikorav that $\widehat{\mathbb{Z}\mathbb{Z}^n}_\xi$ is a Euclidean ring, compare Pajitnov [16, §1]. Therefore $K_1(\widehat{\mathbb{Z}\mathbb{Z}^n}_\xi)$ is given by the group of units. It is easy to see that the group of units in this case is exactly the group factored out in the definition of the Whitehead group of the Novikov ring. Thus $\mathrm{Wh}(\mathbb{Z}^n; \xi) = 0$. Unfortunately this argument does not even generalize to homomorphisms $\xi : \mathbb{Z}^n \to \mathbb{R}$ which are not injective.

## 4. Homomorphism from free abelian groups to the reals

Assume that $G$ is a finitely generated group and $\xi : G \to \mathbb{R}$ a nonzero homomorphism. Then $\xi$ factors through the abelianization of $G$ which is a finitely generated abelian group. Thus $\mathrm{Hom}(G, \mathbb{R})$ is a finite dimensional vector space and has a natural topology. We also define

$$S(G) \quad = \quad \mathrm{Hom}(G, \mathbb{R}) - \{0\}/\sim$$

where $\xi \sim \eta$ means that there is a $c > 0$ such that $\xi = c\eta$. This is a sphere of dimension $\mathrm{rank}(G/[G,G]) - 1$. We will write $[\xi] \in S(G)$ for the equivalence class of a nonzero homomorphism $\xi : G \to \mathbb{R}$.

Now if $\xi : G \to \mathbb{R}$ is a nonzero homomorphism, there exists a unique $n \in \mathbb{Z}$ such that $\xi$ factors as $\bar{\xi} \circ p$ with $p : G \to \mathbb{Z}^n$ surjective and $\bar{\xi} : \mathbb{Z}^n \to \mathbb{R}$ injective. This $n$ is called the rank of $n$. If $\mathrm{rank}\, \xi = 1$, we call $\xi$ *rational*. We also write $S_{\mathbb{Q}}(G)$ for the image of the rational homomorphisms in $S(G)$.

We will now take a closer look at the case $G = \mathbb{Z}^n$.

**Lemma 4.1.** *For every $\xi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{R})$ and a neighborhood $\mathcal{U}$ of $\xi$ there is a rational $\eta \in \mathcal{U}$ with $\ker \xi \subset \ker \eta$. In particular $S_{\mathbb{Q}}(G)$ is dense in $S(G)$ for every finitely generated group $G$.*

*Proof.* We can assume that $\xi$ is injective. Let $e_1, \ldots, e_n$ be a basis of $\mathbb{Z}^n$. Define $\eta : \mathbb{Z}^n \to \mathbb{Q}$ by $\eta(e_i)$ a rational number close to $\xi(e_i)$. By choosing $\eta(e_i)$ close enough to $\xi(e_i)$ we can assure that $\eta \in \mathcal{U}$. Now $\mathrm{im}\, \eta$ is a finitely generated subgroup of $\mathbb{Q}$, hence cyclic. □

**Lemma 4.2.** *Let $\xi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{R}) - \{0\}$ and $\mathcal{U}$ a neighborhood of $\xi$ in $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{R})$. Let $k \geq 1$ be the rank of $\xi$. Then there exist $t_i \in (0,1]$ and rational $\xi_i \in \mathcal{U}$ for $i = 1, \ldots, k$ with*

$$1 = \sum_{i=1}^{k} t_i \quad \textit{and} \quad \xi = \sum_{i=1}^{k} t_i \xi_i.$$

*Proof.* The proof proceeds by induction on $k$. The case $k = 1$ is trivial so we assume $k \geq 2$. Then $\mathrm{im}\, \xi$ is dense in $\mathbb{R}$.

By Lemma 4.1 we can find a rational $\xi_1 \in \mathcal{U}$ such that $\ker \xi \subset \ker \xi_1$. Let $\bar{\xi}, \bar{\xi}_1 : \mathbb{Z}^n / \ker \xi \cong \mathbb{Z}^k \to \mathbb{R}$ be the induced homomorphisms. Let $e_1 \in \mathbb{Z}^k$ be an element with $\xi_1(e_1) > 0$ a generator of the infinite cyclic group $\mathrm{im}\, \xi_1$. Write $\mathbb{Z}^k = \langle e_1 \rangle \oplus \mathbb{Z}^{k-1}$. Let $m$ be a positive integer. Then we can find $x_m \in \mathbb{Z}^{k-1}$ such that $0 < \bar{\xi}(me_1 + x_m)$ is arbitrarily close to 0. Also $\bar{\xi}_1(me_1 + x_m) = m\bar{\xi}_1(e_1)$ can be made arbitrarily large. Choose $t \in (0,1)$ such that $\bar{\xi}(me_1 + x_m) = t\bar{\xi}_1(me_1 + x_m)$. Since $t\xi_1$ is close to $t\xi$, we get that $\xi - t\xi_1$ is close to $(1-t)\xi$. We can assume $t > 0$ to be so small that $\xi - t\xi_1 \in (1-t)\mathcal{U}$. Since $\bar{\xi}(me_1 + x_m) = t\bar{\xi}_1(me_1 + x_m)$ with $me_1 + x_m \neq 0$ we get that $\xi - t\xi_1$ has rank $< k$.

Now let $\mathcal{V} = (1-t)\mathcal{U}$. By induction there exist rational $\xi'_2, \ldots, \xi'_k \in \mathcal{V}$, $t'_2, \ldots, t'_k \in (0,1]$ with $\sum_{i=2}^{k} t'_i = 1$ and

$$\xi - t\xi_1 \quad = \quad \sum_{i=2}^{k} t'_i \xi'_i.$$

Setting $t_1 = t$, $t_i = t'_i(1-t)$ and $\xi_i = \frac{1}{1-t}\xi'_i$ for $i = 2, \ldots, k$ gives the result. □

Lemma 4.2 shows that an injective homomorphism $\xi : \mathbb{Z}^n \to \mathbb{R}$ can be written as a convex combination of $n$ rational homomorphisms which can be chosen arbitrarily close to $\xi$. But we still need to improve on this.

Denote $e_1, \ldots, e_n$ the standard basis of $\mathbb{Z}^n \subset \mathbb{R}^n$ and let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$, that is, the $e_i$ form an orthonormal basis with respect to this inner product.

Now for every homomorphism $\xi : \mathbb{Z}^n \to \mathbb{R}$ there exists a unique vector $v_\xi \in \mathbb{R}^n$ such that $\xi(x) = \langle x, v_\xi \rangle$. For $i = 1, \ldots, n$ let $y_i = \xi(e_i) \in \mathbb{R}$. Then the rank of $\xi$ is equal to the dimension of the $\mathbb{Q}$-subspace of $\mathbb{R}$ generated by the $y_i$. Note that we get a surjective homomorphism $\xi : \mathbb{Z}^n \to \mathbb{Z}$ if and only if all $y_i \in \mathbb{Z}$ and $\gcd(y_1, \ldots, y_n) = 1$.

Assume now that $\xi : \mathbb{Z}^n \to \mathbb{R}$ is injective and let $\mathcal{U}$ be a neighborhood of $[\xi]$ in $S(\mathbb{Z}^n)$. By Lemma 4.2 there exist homomorphisms $\xi_i : \mathbb{Z}^n \to \mathbb{Z}$ and $t_i \in (0, 1]$ for $i = 1, \ldots, n$ with $[\xi_i] \in \mathcal{U}$ and

$$[\xi] \quad = \quad [\sum_{i=1}^n t_i \xi_i].$$

Thus there exist $v_i \in \mathbb{Z}^n$ such that $\xi_i = \langle \cdot, v_i \rangle$ for $i = 1, \ldots, n$ and a $c > 0$ such that $cv_\xi = \sum_{i=1}^n t_i v_i$. Since $\xi$ is injective, we get that $v_1, \ldots, v_n$ is an $\mathbb{R}$-basis of $\mathbb{R}^n$. In general $v_1, \ldots, v_n$ need not be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Now let

$$\Delta(v_1, \ldots, v_n) \quad = \quad \left\{ \sum_{i=1}^n s_i v_i \in \mathbb{R}^n \,|\, 0 \le s_i \le 1 \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=1}^n s_i \le 1 \right\}$$

be the convex hull of the $n + 1$ points $0, v_1, \ldots, v_n$, an $n$-simplex in $\mathbb{R}^n$.

**Lemma 4.3.** *Let $v_1, \ldots, v_n \in \mathbb{Z}^n$ be linearly independent. Then $v_1, \ldots, v_n$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ if and only if*

$$\mathbb{Z}^n \cap \Delta(v_1, \ldots, v_n) \quad = \quad \{0, v_1, \ldots, v_n\}.$$

*Proof.* Assume that $v_1, \ldots, v_n$ is a $\mathbb{Z}$-basis and let $x \in \mathbb{Z}^n \cap \Delta(v_1, \ldots, v_n)$. So there exist $x_i \in \mathbb{Z}$ for $i = 1, \ldots, n$ such that $x = \sum_{i=1}^n x_i \cdot v_i$. Since $x \in \Delta(v_1, \ldots, v_n)$ we must have $0 \le x_i \le 1$ and $\sum_{i=1}^n x_i \le 1$. Thus we can have at most one $x_i = 1$. It follows that $x \in \{0, v_1, \ldots, v_n\}$.

Now assume that $\mathbb{Z}^n \cap \Delta(v_1, \ldots, v_n) = \{0, v_1, \ldots, v_n\}$. Since $v_1, \ldots, v_n$ are linearly independent, they form an $\mathbb{R}$-basis of $\mathbb{R}^n$. Let $x \in \mathbb{Z}^n$. Thus there exist $x_i \in \mathbb{R}$ for $i = 1, \ldots, n$ with $x = \sum_{i=1}^n x_i \cdot v_i$. We can find a $y \in \mathbb{Z}^n$ in the $\mathbb{Z}$-span of $v_1, \ldots, v_n$ such that we have

$$x - y = \sum_{i=1}^n (x_i - y_i) v_i$$

with $0 \le x_i - y_i \le 1$. Without loss of generality we assume $y = 0$.

So $v_1, \ldots, v_n$ is a $\mathbb{Z}$-basis if and only if for every $x = \sum_{i=1}^n x_i \cdot v_i \in \mathbb{Z}^n$ with $0 \le x_i \le 1$ for $i = 1, \ldots, n$ we have $x_i \in \{0, 1\}$ for all $i = 1, \ldots, n$.

Let

$$\Box(v_1, \ldots, v_n) \quad = \quad \left\{ \sum_{i=1}^n s_i v_i \in \mathbb{R}^n \,|\, 0 \le s_i \le 1 \text{ for } i = 1, \ldots, n \right\}.$$

We need to show that

$$\mathbb{Z}^n \cap \square(v_1, \ldots, v_n) \quad = \quad \left\{ \sum_{i=1}^n \delta_i v_i \mid \delta_i \in \{0,1\} \text{ for } i = 1, \ldots, n \right\}. \tag{2}$$

Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map given by $H(e_i) = v_i$ for $i = 1, \ldots, n$. Then $H$ sends $[0,1]^n$ to $\square(v_1, \ldots, v_n)$ and

$$\Delta^n \quad = \quad \left\{ \sum_{i=1}^n s_i e_i \in [0,1]^n \mid \sum_{i=1}^n s_i \leq 1 \right\}$$

to $\Delta(v_1, \ldots, v_n)$.

We claim that $[0,1]^n$ has a triangulation whose 0-simplices is the set $[0,1]^n \cap \mathbb{Z}^n$ and whose $n$-simplices are of the form $K(\Delta^n)$ with $K \in GL(n, \mathbb{Z})$. Then we get a triangulation of $\square(v_1, \ldots, v_n)$ whose set of 0-simplices is the right hand side of (2). Any other element of $\mathbb{Z}^n \cap \square(v_1, \ldots, v_n)$ lies in some $n$-simplex of the form $H(K(\Delta^n))$. Since $K \in GL(n, \mathbb{Z})$ we get an extra element of $\mathbb{Z}^n$ in $H(\Delta^n) = \Delta(v_1, \ldots, v_n)$ which is not possible by assumption. Therefore (2) follows.

It remains to show the triangulation statement, which we will prove by induction.

If $n = 1$ the statement is clear, so assume that $[0,1]^{n-1}$ has a triangulation with 0-simplices the set $[0,1]^{n-1} \cap \mathbb{Z}^{n-1}$ and whose $n-1$-simplices are of the form $K(\Delta^{n-1})$ with $K \in GL(n-1, \mathbb{Z})$.

To get a triangulation of $\Delta^{n-1} \times [0,1]$, look at the triangulation generated by the $n$-simplices $\sigma_j$ for $j = 0, \ldots, n-1$ where $\sigma_j$ has as vertices the points

$$(0,0), (e_1, 0), \ldots, (e_j, 0), (e_j, 1), \ldots, (e_{n-1}, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Rewrite $e_j = (e_j, 0)$ and $e_j + e_n = (e_j, 1)$ for $j = 1, \ldots, n-1$. We also write $e_n = (0,1)$. So $\sigma_j$ has the vertices $0, e_1, \ldots, e_j, e_j + e_n, \ldots, e_{n-1} + e_n$ for $j = 1, \ldots, n-1$ and $\sigma_0$ has the vertices $0, e_n, e_1 + e_n, \ldots, e_{n-1} + e_n$. Clearly there is an $H_j \in GL(n, \mathbb{Z})$ for $j = 0, \ldots, n-1$ such that $H_j(\Delta^n) = \sigma_j$.

The argument can be repeated for $n-1$-simplices of the form $K(\Delta^{n-1})$ with $K \in GL(n, \mathbb{Z})$. Indeed this is triangulated such that the $n$-simplices are of the form $\overline{K}(H_j(\Delta^n))$, where $\overline{K} = i(K)$ with $i : GL(n-1, \mathbb{Z}) \to GL(n, \mathbb{Z})$ the standard inclusion. This finishes the proof of the lemma. $\square$

**Proposition 4.4.** *Let $\xi : \mathbb{Z}^n \to \mathbb{R}$ be an injective homomorphism and $\mathcal{U}$ an open neighborhood of $[\xi] \in S(\mathbb{Z}^n)$. Then there exist homomorphisms $\xi_i : \mathbb{Z}^n \to \mathbb{Z}$ for $i = 1, \ldots, n$ and a $\mathbb{Z}$-basis $t_1, \ldots, t_n$ of $\mathbb{Z}^n$ such that*

1. $[\xi_i] \in \mathcal{U}$ *for all* $i = 1, \ldots, n$.
2. $\bigcap_{i=1}^n \widehat{R\mathbb{Z}^n}_{\xi_i} \subset \widehat{R\mathbb{Z}^n}_\xi$.
3. $\xi_i(t_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$ *for all* $i, j = 1, \ldots, n$.

*Proof.* By Lemma 4.2 there exist homomorphisms $\xi'_i : \mathbb{Z}^n \to \mathbb{Z}$ and $t_i \in (0,1]$ for $i = 1, \ldots, n$ such that $[\xi'_i] \in \mathcal{U}$ and $[\xi] = [\sum t_i \xi'_i]$. Since $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{R})$ is locally convex we can also assume that $[\sum s_i \xi'_i] \in \mathcal{U}$ for every $(s_1, \ldots, s_n) \in [0,1]^n$.

Let $v'_i \in \mathbb{Z}^n$ be such that $\xi'_i(x) = \langle x, v'_i \rangle$ and $v \in \mathbb{R}^n$ such that $\xi(x) = \langle x, v \rangle$. Look at $\Delta(v'_1, \ldots, v'_n)$. Note that the $\mathbb{R}$-subspace $\langle v \rangle$ generated by $v$ has nontrivial intersection with the interior of $\Delta(v'_1,\ldots,v'_n)$. Also, if $y \in \Delta(v'_1,\ldots,v'_n)\cap\mathbb{Z}^n$, then $[\xi_y] \in \mathcal{U}$ where $\xi_y(x) = \langle x, y\rangle$ by the convexity property that we assume.

By compactness of $\Delta(v'_1,\ldots,v'_n)$ the set

$$A \quad = \quad \mathbb{Z}^n \cap \Delta(v'_1,\ldots,v'_n) - \{0, v'_1,\ldots,v'_n\}$$

is finite. Let $B \subset \mathrm{Hom}(\mathbb{Z}^n,\mathbb{R})$ be the ball around 0 of radius 1, that is, $B = \{v \in \mathbb{R}^n \mid \langle v, v\rangle \leq 1\}$.

For $y \in A$ and $j = 1,\ldots,n$ let $\Delta_j = \Delta(y, v_1,\ldots,v'_{j-1}, v'_{j+1},\ldots,v'_n)$, that is, we replace $v'_j$ by $y$. Then we can write

$$B \cap \Delta(v'_1,\ldots,v'_n) \quad = \quad \bigcup_{j=1}^{n} B \cap \Delta_j$$

and $\Delta_j \cap \Delta_i$ has empty interior for $i \neq j$. Since $\xi$ is injective there is a unique $j$ such that $\langle v\rangle \cap \mathrm{int}\Delta_j \neq \emptyset$. We can think of $y, v'_1,\ldots,v'_{j-1}, v'_{j+1},\ldots,v'_n$ giving a better approximation of $v$ than $v'_1,\ldots,v'_n$, compare Figure 1, where $\Delta(x,z)$ should be replaced by $\Delta(x,y)$.

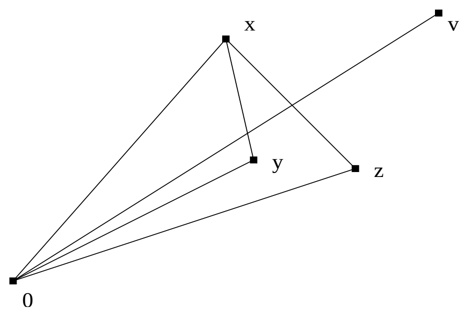


FIGURE 1.

Let

$$A_1 \quad = \quad \mathbb{Z}^n \cap \Delta(y, v'_1,\ldots,v'_{j-1}, v'_{j+1},\ldots,v'_n) - \{0, y, v'_1,\ldots,v'_{j-1}, v'_{j+1},\ldots,v'_n\}$$

for this $j$. Clearly $A_1 \subset A - \{y\}$, so after finitely many steps we get vectors $v_1,\ldots,v_n \in \mathbb{Z}^n$ such that

$$\mathbb{Z}^n \cap \Delta(v_1,\ldots,v_n) \quad = \quad \{0, v_1,\ldots,v_n\}$$

and $\langle v\rangle \cap \mathrm{int}\Delta(v_1,\ldots,v_n) \neq \emptyset$. By Lemma 4.3 we have that $v_1,\ldots,v_n$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

For $i = 1,\ldots,n$ Define $\xi_i : \mathbb{Z}^n \to \mathbb{Z}$ by $\xi_i(x) = \langle x, v_i\rangle$. Then $[\xi_i] \in \mathcal{U}$ and $[\xi] = [\sum s_i\xi_i]$ for some $s_1,\ldots,s_n \in (0,1]$. Therefore we get (1), and (2) by Lemma 2.2.1.

Let $T : \mathbb{Z}^n \to \mathbb{Z}^n$ be the linear map given by $T(v_i) = e_i$ for $i = 1,\ldots,n$. Define the inner product $(x,y) = \langle Tx, Ty\rangle$ and let $T^* : \mathbb{Z}^n \to \mathbb{Z}^n$ be the adjoint of $T$ with respect to $(\cdot,\cdot)$. Note that $v_1,\ldots,v_n$ is an orthonormal basis with respect to this inner product. Now let $t_i = TT^*v_i$ for $i = 1,\ldots,n$. Then $t_1,\ldots,t_n$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ and

$$\xi_i(t_j) = \langle TT^*v_j, v_i\rangle = (T^*v_j, T^{-1}v_i) = (v_j, v_i) = \delta_{ij}.$$

This finishes the proof. $\square$

## 5. Proof of Theorem 3.2

**Lemma 5.1.** *Let $\xi : G \to \mathbb{R}$ be a nonzero homomorphism and $\tau_0 \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi$. Then there exists a matrix $A$ over $RG$ which is invertible over $\widehat{RG}_\xi$ with $\tau(A) = \tau_0 \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi$. Furthermore, if $G$ is finitely generated, there is a neighborhood $\mathcal{U}$ of $[\xi]$ in $S(G)$ such that $A$ is invertible over $\bigcap_{[\eta]\in\mathcal{V}} \widehat{RG}_\eta$ for every subset $\mathcal{V} \subset \mathcal{U}$.*

*Proof.* Let $\bar{A}$ be an invertible $n \times n$ matrix over $\widehat{RG}_\xi$ with $\tau(\bar{A}) = \tau_0$. Let $\bar{A}^{-1}$ be its inverse. Choose a matrix $A$ over $RG$ such that $\|A - \bar{A}\|_\xi < \min\{1, \|\bar{A}^{-1}\|_\xi^{-1}\}$ and a matrix $B$ over $RG$ such that $\|B - \bar{A}^{-1}\|_\xi < \min\{1, \|\bar{A}\|_\xi^{-1}\}$. To do this define

$$A_{ij}(g) = \begin{cases} \bar{A}_{ij}(g) & \text{for } \exp(\xi(g)) \geq \min\{1, \|\bar{A}^{-1}\|_\xi^{-1}\} \\ 0 & \text{otherwise} \end{cases}$$

and similarly for $B$. Then

$$\begin{array}{rclcl} A \cdot B & = & (\bar{A} + (A - \bar{A})) \cdot (\bar{A}^{-1} + (B - \bar{A}^{-1})) & = & I - C \\ B \cdot A & = & (\bar{A}^{-1} + (B - \bar{A}^{-1})) \cdot (\bar{A} + (A - \bar{A})) & = & I - C' \end{array}$$

with $\|C\|_\xi, \|C'\|_\xi < 1$. Since $A$ and $B$ are matrices over $RG$, so are $C$ and $C'$. Also there is an $\varepsilon > 0$ such that $\|C\|_\xi, \|C'\|_\xi \leq 1 - \varepsilon$. Let

$$F = \bigcup_{i,j=1}^{n} \operatorname{supp} C_{ij} \cup \operatorname{supp} C'_{ij},$$

a finite subset of $G$. In particular $\xi(g) < 0$ for all $g \in F$. There is a neighborhood $\mathcal{U}'$ of $\xi$ in $\operatorname{Hom}(G, \mathbb{R})$ such that $\eta(g) < 0$ for every $g \in F$ and every $\eta \in \mathcal{U}'$. Let $\mathcal{U}$ be the projection of $\mathcal{U}'$ to $S(G)$. Then $\|C\|_\eta, \|C'\|_\eta < 1$ for every $\eta \in \mathcal{U}$ and we get that $I - C$ is invertible over $\widehat{RG}_\eta$ with inverse $I + C + C^2 + \dots$ and the same for $I - C'$. Then $A$ has a left and a right inverse over intersections of such Novikov rings.

To see that $\tau(A) = \tau(\bar{A}) \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi$ note that

$$A \cdot \bar{A}^{-1} = (\bar{A} + (A - \bar{A})) \cdot \bar{A}^{-1} = I - D$$

with $\|D\|_\xi < 1$. $\square$

Now assume that $G$ is finitely generated, so that there is a $k \geq 1$ such that $G/\ker\xi \cong \mathbb{Z}^k$. Now let $\mathcal{U}$ be neighborhood of $[\xi]$ in $S(G)$. By Proposition 4.4 we can find homomorphisms $\xi_i : G \to \mathbb{Z}$ for $i = 1, \dots, k$ with $[\xi_i] \in \mathcal{U}$, $\bigcap_{i=1}^k \widehat{RG}_{\xi_i} \subset \widehat{RG}_\xi$, and $g_1, \dots, g_k \in G$ such that $\xi_i(g_j) = -\delta_{ij}$ for $i, j = 1, \dots, k$. Picking $g_i$ with $\xi_i(g_i) = -1$ instead of $+1$ has mainly cosmetic purposes.

For $j = 0, \dots, k$ let

$$\begin{array}{rcl} \Lambda_j & = & \widehat{RG}^o_{\xi_1} \cap \dots \cap \widehat{RG}^o_{\xi_j} \cap \widehat{RG}_{\xi_{j+1}} \cap \dots \cap \widehat{RG}_{\xi_k} \\ & = & \left\{ \lambda \in \widehat{RG}_{\xi_1} \cap \dots \cap \widehat{RG}_{\xi_k} \mid \|\lambda\|_{\xi_i} \leq 1 \text{ for } i = 1, \dots, j \right\} \end{array}$$

Note that $\Lambda_0 = \bigcap_{i=1}^k \widehat{RG}_{\xi_i}$ and that the ring $\Lambda_j$ is obtained from $\Lambda_{j+1}$ by inverting $g_{j+1}$.

Also define for $j = 1, \ldots, k$

$$\begin{aligned} G_j &= \{g \in G \,|\, \xi_i(g) \leq 0 \text{ for } i \leq j\} \\ K_j &= \{g \in G_j \,|\, \xi_j(g) = 0\} \end{aligned}$$

We then have subrings $RK_j \subset RG_j \subset \Lambda_j$ for $j = 1, \ldots, k$.

Denote $i_* : K_1(\Lambda_j) \to K_1(\Lambda_0)$ and $i_* : K_1(RG) \to K_1(\Lambda_0)$ the natural maps.

**Proposition 5.2.** *Let $n$ be a positive integer and $A : (\Lambda_j)^n \to (\Lambda_j)^n$ an automorphism for some $j \in \{0, \ldots, k-1\}$. Then there exist $\tau_1 \in K_1(RG)$ and $\tau_2 \in K_1(\Lambda_{j+1})$ with*

$$i_*\tau(A) \;=\; i_*(\tau_1) + i_*(\tau_2) \in K_1(\Lambda_0).$$

The proof of this proposition uses the methods of Pajitnov and Ranicki [20, Lm.2.18-2.19]. Since our notation differs quite a bit from theirs we give a full proof, but defer it to the next section. Assuming Proposition 5.2 we can now proof Theorem 3.2.

*Proof of Theorem 3.2.* Assume $G$ is finitely generated. Let $\tau_0 \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi$. We can represent $\tau_0$ by an invertible matrix $A$. By Lemma 5.1 we can assume that $A$ has entries in $RG$ and that there is a neighborhood $\mathcal{U}$ of $\xi$ such that $A$ is invertible over $\bigcap_{\eta \in \mathcal{V}} \widehat{RG}_\eta$ for every subset $\mathcal{V} \subset \mathcal{U}$.

Choose the $\xi_i$ as above so we get that $A$ is invertible over $\Lambda_0$. In particular we get $\tau_0 = i_*\tau(A)$ where $i_* : K_1(\Lambda_0) \to K_1(\widehat{RG}_\xi)$ is induced by the inclusion of Lemma 2.2 (1).

Iterating Proposition 5.2 we get

$$\tau_0 \;=\; i_*(\tau_k) + i_*(\tau') \tag{3}$$

with $\tau_k \in K_1(\Lambda_k)$ and $\tau' \in K_1(RG)$. But the inclusion $\Lambda_k \subset \widehat{RG}_\xi$ factors through $\widehat{RG}^o_\xi$ by Lemma 2.2 (2) and therefore we get

$$i_*(\tau_k) \;=\; i_*(\tau(w)) + i_*(\tau'') \in K_1(\widehat{RG}_\xi) \tag{4}$$

with $\tau(w) \in W_\xi$ and $\tau'' \in K_1(RG)$ by Proposition 3.1. But $i_*(\tau(w)) \in \overline{W}_\xi$ so by combining (3) and (4) we get $\tau_0 = i_*(\tau' + \tau'') \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi$ with $\tau' + \tau'' \in K_1(RG)$. This finishes the proof for finitely generated $G$.

For the general case we need two more lemmas.

**Lemma 5.3.** *Let $A$ be an invertible $n \times n$ matrix over $\widehat{RG}_\xi$ with $\tau(A) = 0 \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi$. Then there exist elementary matrices $E_1, \ldots, E_k$ over $RG$ and a matrix $E$ over $\widehat{RG}_\xi$ with $\|E\|_\xi < 1$ such that for a stabilization of $A$ we get*

$$\begin{pmatrix} A & \\ & I \end{pmatrix} \;=\; E_1 \cdots E_k \cdot (I - E)$$

*Proof.* Since $i_*\tau(A) = 0$ we get $\begin{pmatrix} A & \\ & I \end{pmatrix} = F_1 \cdots F_l$ with the $F_i$ being either elementary matrices over $\widehat{RG}_\xi$ or matrices of the form $I - D$ with $\|D\|_\xi < 1$. Since the elementary matrices generate the commutator of $\mathrm{GL}(R)$ for any ring $R$ with unit we can assume that $F_l = I - D$ with $\|D\|_\xi < 1$ and the remaining matrices are elementary.

It remains to show that we can replace the elementary matrices over $\widehat{RG}_\xi$ by elementary matrices over $RG$. For this we will prove the following:

Given elementary matrices $E_1', \ldots, E_k'$ over $\widehat{RG}_\xi$ and $\varepsilon \in (0,1)$, there exist elementary matrices $E_1, \ldots, E_k$ over $RG$ and a matrix $E$ over $RG$ with $\|E\|_\xi < \varepsilon$, such that

$$E_1' \cdots E_k' \quad = \quad E_1 \cdots E_k \cdot (I - E) \tag{5}$$

We prove it by induction on $k$. The case $k = 0$ is trivial. Now assume the statement is true for $k-1$. Then $E_1' \cdots E_k' = E_1' \cdots E_{k-1}' \cdot E_k'$. By induction hypothesis we can find elementary matrices $E_1, \ldots, E_{k-1}$ over $RG$ and $E'$ with $\|E'\|_\xi < \varepsilon \cdot \|E_k'\|_\xi^{-2}$ such that $E_1' \cdots E_{k-1}' = E_1 \cdots E_{k-1} \cdot (I - E')$. Now

$$(I - E') \cdot E_k' \quad = \quad E_k' \cdot (I - (E_k')^{-1} \cdot E' \cdot E_k').$$

Since we can write $E_k' = E_k - R_k = E_k(I - E_k^{-1} R_k)$ with $E_k$ an elementary matrix over $RG$ and $\|R_k\|_\xi < \varepsilon \cdot \|E_k'\|_\xi^{-1}$ we get the claim. Notice that $\|E_k'\|_\xi^{-1} = \|E_k\|_\xi^{-1} = \|E_k^{-1}\|_\xi^{-1}$ and $\|F\|_\xi \geq 1$ for every elementary matrix $F$.

This shows (5) and the lemma follows. □

If $H \leq G$ is a finitely generated subgroup, we get a subring $\widehat{RH}_\xi \subset \widehat{RG}_\xi$ and an induced map $i_* : K_1(\widehat{RH}_\xi)/\overline{W}_\xi(H) \to K_1(\widehat{RG}_\xi)/\overline{W}_\xi(G)$. Furthermore we get a direct system $(H_j)_{j \in I}$ of finitely generated subgroups of $G$ ordered by inclusion which induces a direct system of abelian groups $\Big(K_1(\widehat{RH_j}_\xi)/\overline{W}_\xi(H_j)\Big)_{j \in I}$.

**Lemma 5.4.** *Let $G$ be a group and $\xi : G \to \mathbb{R}$ a homomorphism. Then $K_1(\widehat{RG}_\xi)/\overline{W}_\xi(G)$ is the direct limit of $\Big(K_1(\widehat{RH_j}_\xi)/\overline{W}_\xi(H_j)\Big)_{j \in I}$, where $(H_j)_{j \in I}$ are the finitely generated subgroups of $G$.*

*Proof.* We need to show that

1. for every $\tau_0 \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi(G)$ there is a finitely generated subgroup $H$ and $\tau' \in K_1(\widehat{RH}_\xi)/\overline{W}_\xi(H)$ with $\tau_0 = i_* \tau'$.
2. If $\tau_0 \in K_1(\widehat{RH_1}_\xi)/\overline{W}_\xi(H_1)$ satisfies $i_* \tau_0 = 0 \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi(G)$ for a finitely generated $H_1$, there exists a finitely generated subgroup $H_2$ containing $H_1$ such that $i_* \tau_0 = 0 \in K_1(\widehat{RH_2}_\xi)/\overline{W}_\xi(H_2)$.

For (1) represent $\tau_0$ by an invertible matrix $\bar{A}$ over $\widehat{RG}_\xi$. Choose matrices $A$, $B$ over $RG$ with $\|A - \bar{A}\|_\xi < \min\{1, \|\bar{A}^{-1}\|_\xi^{-1}\}$ and $\|B - \bar{A}^{-1}\|_\xi < \min\{1, \|\bar{A}\|_\xi^{-1}\}$. Then $A \cdot B = I - C$ with $\|C\|_\xi < 1$ and $A$ is invertible with $A^{-1} = B \cdot (I - C)^{-1}$. Also $C = I - A \cdot B$ is a matrix over $RG$. Hence

$$F \quad = \quad \bigcup_{i,j=1}^{n} \operatorname{supp} A_{ij} \cup \operatorname{supp} B_{ij} \cup \operatorname{supp} C_{ij}$$

is a finite subset of $G$ which generates a finitely generated subgroup $H$. Also $B \cdot (I - C)^{-1}$ is a well defined matrix over $\widehat{RH}_\xi$ and we get $\tau_0 = i_* \tau(A)$.

Now let $A$ be an invertible matrix over $\widehat{RH_1}_\xi$ with $i_*\tau(A) = 0 \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi(G)$. By Lemma 5.3 we get

$$\begin{pmatrix} A & \\ & I \end{pmatrix} = E_1 \cdots E_k \cdot (I - E)$$

with $E_i$ elementary matrices over $RG$ and $\|E\|_\xi < 1$. Let

$$F = \bigcup_{i,j=1}^{n} \bigcup_{l=1}^{k} \operatorname{supp}(E_l)_{ij},$$

a finite subset of $G$, and let $H_2$ be the subgroup of $G$ generated by $H_1$ and $F$, a finitely generated subgroup of $G$. As above it follows that $I - E$ is an invertible matrix over $\widehat{RH_2}_\xi$ and we get $i_*\tau(A) = 0 \in K_1(\widehat{RH_2}_\xi)/\overline{W}_\xi(H_2)$. □

We note that Lemma 5.4 is not true in general if we replace $K_1(\widehat{RG}_\xi)/\overline{W}_\xi(G)$ by $K_1(\widehat{RG}_\xi)$.

For a finitely generated subgroup $H$ of $G$ we already know that $i_* : K_1(RH) \to K_1(\widehat{RH}_\xi)/\overline{W}_\xi(H)$ is surjective. Thus we get a surjection of direct systems

$$\Big(i_* : K_1(RH_j) \to K_1(\widehat{RH_j}_\xi)/\overline{W}_\xi(H_j)\Big)_{j \in I}.$$

Since the direct limit is an exact functor we get a surjection between the direct limits. By Lemma 5.4 this means we get a surjection $i_* : K_1(RG) \to K_1(\widehat{RG}_\xi)/\overline{W}_\xi(G)$ which is clearly the map in Theorem 3.2. □

## 6. Proof of Proposition 5.2

We keep the notation established above Proposition 5.2. We will frequently write $\Lambda_j^n$ for the finitely generated free $\Lambda_j$-module $(\Lambda_j)^n$. Similarly we will write $g_j^l$ for $(g_j)^l$, where $l$ is an integer.

Recall that $\xi_{j+1}(g_{j+1}) = -1$, so $g_{j+1}$ defines a left $\Lambda_{j+1}$-module morphism $g_{j+1} : \Lambda_{j+1} \to \Lambda_{j+1}$ by $x \mapsto x \cdot g_{j+1}$.

**Lemma 6.1.** *Let $l$ be a positive integer. Then the $\Lambda_{j+1}$-module morphism $g_{j+1}^l : \Lambda_{j+1}^n \to \Lambda_{j+1}^n$, $x \mapsto x \cdot g_{j+1}^l$ is such that $\operatorname{coker} g_{j+1}^l$ is a finitely generated free $RK_{j+1}$-module.*

*Proof.* It suffices to look at the case $n, l = 1$. Let $x \in \Lambda_{j+1}$. If $g \in \operatorname{supp} x$, then $\xi_i(g) \le 0$ for $i \le j+1$. If $\xi_{j+1}(g) < 0$, then $g \cdot g_{j+1}^{-1} \in \Lambda_{j+1}$. Hence we can write $x = x_1 + x_2$ with $x_1 \in RK_{j+1}$ and $x_2 \cdot g_{j+1}^{-1} \in \Lambda_{j+1}$, and this decomposition is unique. But $x_2 \in \operatorname{im} g_{j+1}$ and so $\operatorname{coker} g_{j+1} = RK_{j+1}$. □

We have that $A : \Lambda_j^n \to \Lambda_j^n$ is an automorphism. Choose $l \ge 0$ so that for $x \in \Lambda_{j+1}^n$ we get $A(x) \cdot g_{j+1}^l \in \Lambda_{j+1}^n \subset \Lambda_j^n$. Then we can define an injective $\Lambda_{j+1}$-module morphism

$$\begin{aligned} \tilde{A} : \Lambda_{j+1}^n &\longrightarrow \Lambda_{j+1}^n \\ x &\mapsto A(x) \cdot g_{j+1}^l \end{aligned}$$

Let

$$P_{j+1} = \operatorname{coker}(\tilde{A} : \Lambda_{j+1}^n \to \Lambda_{j+1}^n).$$

The next lemma is the analogue of Pajitnov and Ranicki [20, Lm.2.18].

**Lemma 6.2.** *We have*

(1) *$P_{j+1}$ is a finitely generated projective $RK_{j+1}$-module.*
(2) *The map $\nu : P_{j+1} \to P_{j+1}$, $x \mapsto g_{j+1} \cdot x$ is nilpotent.*

*Proof.* Let $B : \Lambda_j^n \to \Lambda_j^n$ be the inverse of $A$. Choose $m \geq 0$ so that for all $x \in \Lambda_{j+1}^n$ we get $B(x \cdot g_{j+1}^{-l}) \cdot g_{j+1}^m \in \Lambda_{j+1}^n \subset \Lambda_j^n$. Define the $\Lambda_{j+1}$-module morphism

$$\begin{array}{rcl} \tilde{B} : \Lambda_{j+1}^n & \longrightarrow & \Lambda_{j+1}^n \\ x & \mapsto & B(x \cdot g_{j+1}^{-l}) \cdot g_{j+1}^m \end{array}$$

Restriction defines an $RK_{j+1}$-module morphism $r : \Lambda_j^n \to \Lambda_{j+1}^n$ with the property that $r \circ i = \mathrm{id} : \Lambda_{j+1}^n \to \Lambda_{j+1}^n$. Thus define the $RK_{j+1}$-module morphism

$$\begin{array}{rcl} \tilde{C} : \Lambda_{j+1}^n & \longrightarrow & \Lambda_{j+1}^n \\ x & \mapsto & r(A(x \cdot g_{j+1}^{-m}) \cdot g_{j+1}^l) \end{array}$$

We get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Lambda_{j+1}^n \arrow[r, "\tilde{A}"] \arrow[d, equal] & \Lambda_{j+1}^n \arrow[r] \arrow[d, "\tilde{B}"] & P_{j+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Lambda_{j+1}^n \arrow[r, "g_{j+1}^m"] \arrow[d, equal] & \Lambda_{j+1}^n \arrow[r] \arrow[d, "\tilde{C}"] & \bigoplus_{s=1}^m RK_{j+1}^n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Lambda_{j+1}^n \arrow[r, "\tilde{A}"] & \Lambda_{j+1}^n \arrow[r] & P_{j+1} \arrow[r] & 0 \end{tikzcd}$$

It is easy to see that $\tilde{C} \circ \tilde{B} = \mathrm{id} : \Lambda_{j+1}^n \to \Lambda_{j+1}^n$ and therefore $P_{j+1}$ is finitely generated projective over $RK_{j+1}$ as a direct summand of a finitely generated free $RK_{j+1}$-module. Here the middle row follows from Lemma 6.1.

To see that $\nu$ is nilpotent, let $x \in \Lambda_{j+1}^n$. In $\Lambda_j^n$ we get

$$\begin{array}{rcl} g_{j+1}^{m+l} \cdot x & = & g_{j+1}^{m+l} \cdot x \cdot g_{j+1}^{-l} \cdot g_{j+1}^l \;=\; A \circ B(g_{j+1}^{m+l} \cdot x \cdot g_{j+1}^{-l}) \cdot g_{j+1}^l \\ & = & A(g_{j+1}^m \cdot B(g_{j+1}^l \cdot x \cdot g_{j+1}^{-l}) \cdot g_{j+1}^m \cdot g_{j+1}^{-m}) \cdot g^l \\ & = & A(g_{j+1}^m \cdot \tilde{B}(g_{j+1}^l \cdot x) \cdot g_{j+1}^{-m}) \cdot g_{j+1}^l \;=\; \tilde{A}(y) \end{array}$$

with $y = g_{j+1}^m \cdot \tilde{B}(g_{j+1}^l \cdot x) \cdot g_{j+1}^{-m} \in \Lambda_{j+1}^n$. Thus $g_{j+1}^{m+l} \cdot x \in \mathrm{im}\, \tilde{A}$. □

We have that $P_{j+1}$ is also a $\Lambda_{j+1}$-module. Define a $\Lambda_{j+1}$-module morphism

$$\begin{array}{rcl} \pi : \Lambda_{j+1} \otimes_{RK_{j+1}} P_{j+1} & \longrightarrow & P_{j+1} \\ \lambda \otimes x & \mapsto & \lambda \cdot x \end{array}$$

Let

$$\Lambda_{j+1} g_{j+1} \quad = \quad \{\lambda g_{j+1} \in \Lambda_{j+1} \,|\, \lambda \in \Lambda_{j+1}\}$$

Then $(\Lambda_{j+1} g_{j+1})^n$ is a free $\Lambda_{j+1}$-module. Also $RK_{j+1}$ acts on the right by ordinary multiplication. Notice that if we write $\lambda g_{j+1}$ for the elements of $\Lambda_{j+1} g_{j+1}$ this means $\lambda g_{j+1}\cdot r = \lambda(g_{j+1} r g_{j+1}^{-1})g_{j+1}$ for $r \in RK_{j+1}$. Define the $\Lambda_{j+1}$-module morphism

$$\begin{array}{rcl}\rho : \Lambda_{j+1}g_{j+1}\otimes_{RK_{j+1}} P_{j+1} & \longrightarrow & \Lambda_{j+1}\otimes_{RK_{j+1}} P_{j+1}\\ \lambda g_{j+1}\otimes x & \mapsto & \lambda g_{j+1}\otimes x - \lambda\otimes g_{j+1}\cdot x\end{array}$$

**Lemma 6.3.** *The following sequence is a finitely generated projective $\Lambda_{j+1}$-module resolution of $P_{j+1}$.*

$$0 \longrightarrow \Lambda_{j+1}g_{j+1}\otimes_{RK_{j+1}} P_{j+1} \xrightarrow{\rho} \Lambda_{j+1}\otimes_{RK_{j+1}} P_{j+1} \xrightarrow{\pi} P_{j+1} \longrightarrow 0$$

*Proof.* We can split the sequence over $RK_{j+1}$ using the $RK_{j+1}$-module morphisms

$$\begin{array}{rcl}\sigma : P_{j+1} & \longrightarrow & \Lambda_{j+1}\otimes_{RK_{j+1}} P_{j+1}\\ x & \mapsto & 1\otimes x\end{array}$$

and

$$\begin{array}{rcl}\omega : \Lambda_{j+1}\otimes_{RK_{j+1}} P_{j+1} & \longrightarrow & \Lambda_{j+1}g_{j+1}\otimes_{RK_{j+1}} P_{j+1}\\ \lambda\otimes x & \mapsto & \overline{\lambda}\otimes x + \overline{\lambda g_{j+1}^{-1}}\otimes g_{j+1}x + \overline{\lambda g_{j+1}^{-2}}\otimes g_{j+1}^2 x + \dots\end{array}$$

where $\overline{\cdot} : \Lambda_j \to \Lambda_{j+1}g_{j+1}$ denotes restriction. Notice that we have a finite sum only, since $g_{j+1}^{m+l}\cdot x = 0$ by Lemma 6.2 (2). This shows that the sequence is exact. $\square$

The two projective $\Lambda_{j+1}$ resolutions can be related by a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_{j+1}^n \arrow[r, "\tilde{A}"] \arrow[d, "f"] & \Lambda_{j+1}^n \arrow[r] \arrow[d, "g"] & P_{j+1} \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & \Lambda_{j+1}g_{j+1}\otimes_{RK_{j+1}} P_{j+1} \arrow[r, "\rho"] & \Lambda_{j+1}\otimes_{RK_{j+1}} P_{j+1} \arrow[r, "\pi"] & P_{j+1} \arrow[r] & 0
\end{tikzcd}$$

We can think of $(f, g)$ as a chain homotopy equivalence between 1-dimensional finitely generated projective $\Lambda_{j+1}$-chain complexes. Notice that after tensoring with $\Lambda_0$ we get that both $1\otimes\tilde{A}$ and $1\otimes\rho$ become automorphisms, since

$$\begin{array}{rcl}\Lambda_0\otimes_{RK_{j+1}} P_{j+1} & \longrightarrow & \Lambda_0\otimes_{\Lambda_{j+1}}\Lambda_{j+1}g_{j+1}\otimes_{RK_{j+1}} P_{j+1}\\ \lambda\otimes p & \mapsto & \lambda g_{j+1}^{-1}\otimes g_{j+1}\otimes p\end{array}$$

is a canonical isomorphism.

The sequence

$$0 \longrightarrow \Lambda_{j+1}^n \xrightarrow{\begin{pmatrix} f \\ \tilde{A}\end{pmatrix}} \Lambda_{j+1}g_{j+1}\otimes_{RK_{j+1}} P_{j+1}\oplus\Lambda_{j+1}^n$$

$$\xrightarrow{\begin{pmatrix}\rho & -g\end{pmatrix}} \Lambda_{j+1}\otimes_{RK_{j+1}} P_{j+1} \longrightarrow 0$$

splits, so denote $\begin{pmatrix} d_1 & d_2\end{pmatrix} : \Lambda_{j+1}g_{j+1}\otimes_{RK_{j+1}} P_{j+1}\oplus\Lambda_{j+1}^n \to \Lambda_{j+1}^n$ a morphism with $d_1 f + d_2\tilde{A} = \mathrm{id}_{\Lambda_{j+1}^n}$. Denote

$$h = \begin{pmatrix}\rho & -g\\ d_1 & d_2\end{pmatrix} : \Lambda_{j+1}g_{j+1}\otimes_{RK_{j+1}} P_{j+1}\oplus\Lambda_{j+1}^n \to \Lambda_{j+1}\otimes_{RK_{j+1}} P_{j+1}\oplus\Lambda_{j+1}^n$$

the resulting isomorphism. Restriction defines a ring homomorphism $T_{j+1} : \Lambda_{j+1} \to RK_{j+1}$ such that $T_{j+1} \circ i : RK_{j+1} \to RK_{j+1}$ is the identity. We get an isomorphism

$$(i \circ T_{j+1})_* h : \Lambda_{j+1} g_{j+1} \otimes_{RK_{j+1}} P_{j+1} \oplus \Lambda_{j+1}^n \to \Lambda_{j+1} \otimes_{RK_{j+1}} P_{j+1} \oplus \Lambda_{j+1}^n$$

since $\Lambda_{j+1} \otimes_{RK_{j+1}} RK_{j+1} \otimes_{\Lambda_{j+1}} \Lambda_{j+1} g_{j+1} \otimes_{RK_{j+1}} P_{j+1} = \Lambda_{j+1} g_{j+1} \otimes_{RK_{j+1}} P_{j+1}$. Therefore we get an automorphism

$$h \circ ((i \circ T_{j+1})_* h)^{-1} : \Lambda_{j+1} \otimes_{RK_{j+1}} P_{j+1} \oplus \Lambda_{j+1}^n \to \Lambda_{j+1} \otimes_{RK_{j+1}} P_{j+1} \oplus \Lambda_{j+1}^n$$

which defines a torsion

$$\tau(f, g) \in K_1(\Lambda_{j+1}).$$

Since

$$\begin{array}{ccc} RG \otimes_{RK_{j+1}} P_{j+1} & \longrightarrow & RG \otimes_{RK_{j+1}} RK_{j+1} \otimes_{\Lambda_{j+1}} \Lambda_{j+1} g_{j+1} \otimes_{RK_{j+1}} P_{j+1} \\ x \otimes p & \mapsto & x g_{j+1}^{-1} \otimes 1 \otimes g_{j+1} \otimes p \end{array}$$

is a canonical isomorphism, we get an automorphism

$$(i_G \circ T_{j+1})_* h : RG \otimes_{RK_{j+1}} P_{j+1} \to RG \otimes_{RK_{j+1}} P_{j+1}$$

where $i_G : RK_{j+1} \to RG$ denotes inclusion. It follows that

$$i_* \tau(f, g) + i_* \tau((i_G \circ T_{j+1})_* h) \quad = \quad \tau(1_{\Lambda_0} \otimes h) \in K_1(\Lambda_0). \tag{6}$$

Note that $\Lambda_0 \otimes_{RK_{j+1}} P_{j+1}$ is canonically isomorphic to $\Lambda_0 \otimes_{RK_{j+1}} RK_{j+1} \otimes_{\Lambda_{j+1}} \Lambda_{j+1} g_{j+1} \otimes_{RK_{j+1}} P_{j+1}$, so $1_{\Lambda_0} \otimes h$ defines an automorphism.

But over $\Lambda_0$ we have the commutative diagram

$$\begin{tikzcd}[column sep=huge] \Lambda_0^n \arrow[r, "\begin{pmatrix} f\tilde{A}^{-1} \\ 1 \end{pmatrix}"] \arrow[d, "\tilde{A}^{-1}"] & \Lambda_0 \otimes_{RK_{j+1}} P_{j+1} \oplus \Lambda_0^n \arrow[r, "\begin{pmatrix} 1 & -f\tilde{A}^{-1} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \rho & -g \\ d_1 & d_2 \end{pmatrix}"] & \Lambda_0 \otimes_{RK_{j+1}} P_{j+1} \arrow[d, "\rho"] \\ \Lambda_0^n \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & \Lambda_0 \otimes_{RK_{j+1}} P_{j+1} \oplus \Lambda_0^n \arrow[r, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & \Lambda_0 \otimes_{RK_{j+1}} P_{j+1} \end{tikzcd}$$

where we have written $\varphi$ instead of $1 \otimes \varphi$ for all the morphisms involved. Since all vertical arrows are automorphisms and the rows are short exact sequences we get

$$\tau(1 \otimes h) \quad = \quad \tau(1 \otimes \rho) - \tau(1 \otimes \tilde{A}) \in K_1(\Lambda_0). \tag{7}$$

Now

$$\tau(1 \otimes \tilde{A}) \quad = \quad i_* \tau(A) + \tau(g_{j+1}^{ln}) \tag{8}$$

and

$$\tau(1 \otimes \rho) \quad = \quad i_* \tau(1 - p) \tag{9}$$

where

$$\begin{array}{rcl} 1 - p : RG \otimes_{RK_{j+1}} P_{j+1} & \longrightarrow & RG \otimes_{RK_{j+1}} P_{j+1} \\ g \otimes x & \mapsto & g \otimes x - g \cdot g_{j+1}^{-1} \otimes g_{j+1} \cdot x \end{array}$$

is an automorphism with inverse $1 + p + p^2 + \ldots + p^{m+l-1}$. Combining (6), (7), (8) and (9) finishes the proof of Proposition 5.2.

## 7. Further remarks and questions

In the case of a rational homomorphism $\xi : G \to \mathbb{R}$ we get a short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 0$$

with $H = \ker \xi$. In that case $RG$ can be identified with a twisted Laurent polynomial ring $RH_\rho[t, t^{-1}]$ where $\rho : RH \to RH$ is an automorphism induced by the action of $\mathbb{Z}$ on $H$. Similarly $\widehat{RG}_\xi$ can be identified with a twisted Laurent series ring

$$RH_\rho((t)) \quad = \quad RH_\rho[[t]][t^{-1}].$$

The classical Bass-Heller-Swan decomposition in the twisted case, see Farrell and Hsiang [10], Siebenmann [28] and Pajitnov and Ranicki [20], then reads

$$K_1(RH_\rho[t,t^{-1}]) \quad = \quad K_1(RH,\rho) \oplus \widetilde{\mathrm{Nil}}_0(RH,\rho) \oplus \widetilde{\mathrm{Nil}}_0(RH,\rho^{-1}) \tag{10}$$

where $\widetilde{\mathrm{Nil}}_0(RH,\rho^{\pm 1})$ is the reduced class group of pairs $(P,\nu)$ with $P$ a finitely generated projective $RH$-module and $\nu : P \to P$ a nilpotent $\rho^{\pm 1}$-endomorphism. Also $K_1(RH,\rho)$ fits into an exact sequence

$$K_1(RH) \xrightarrow{1-\rho} K_1(RH) \xrightarrow{i} K_1(RH,\rho) \xrightarrow{j} K_0(RH) \xrightarrow{1-\rho} K_0(RH)\ .$$

Pajitnov and Ranicki [20] obtained the corresponding decomposition for the Novikov ring which is

$$K_1(RH_\rho((t))) \quad = \quad K_1(RH,\rho) \oplus W_\xi \oplus \widetilde{\mathrm{Nil}}_0(RH,\rho^{-1}). \tag{11}$$

The two decompositions are related in that the natural map $i_* : K_1(RH_\rho[t,t^{-1}]) \to K_1(RH_\rho((t)))$ maps the copy of $\widetilde{\mathrm{Nil}}_0(RH,\rho)$ into $W_\xi$ and is the identity on the remaining direct summands. In particular this implies Theorem 3.2 in the case of a rational homomorphism. It also shows that $i_* : K_1(RG) \to K_1(\widehat{RG}_\xi)/\overline{W}_\xi$ is not an isomorphism in general. But it follows that the diagonal map induced by inclusion

$$\Delta : K_1(RH_\rho[t,t^{-1}]) \longrightarrow K_1(RH_\rho((t))) \oplus K_1(RH_\rho((t^{-1})))$$

is injective. The analogous result for an arbitrary homomorphism $\xi$ also holds.

**Theorem 7.1.** *Let $\xi : G \to \mathbb{R}$ be a nonzero homomorphism. Then the diagonal map*

$$\Delta : K_1(RG) \longrightarrow K_1(\widehat{RG}_\xi)/\overline{W}_\xi \oplus K_1(\widehat{RG}_{-\xi})/\overline{W}_{-\xi},$$

*induced by inclusion, is injective.*

*Proof.* It is enough to consider the case when $G$ is finitely generated. Let $\tau_0 \in K_1(RG)$ satisfy $\Delta(\tau_0) = 0$. Let $A$ be an invertible matrix over $RG$ with $\tau(A) = \tau_0$. In particular $i_*\tau(A) = 0 \in K_1(\widehat{RG}_\xi)/\overline{W}_\xi$. By Lemma 5.3 there exist elementary matrices $E_1, \ldots, E_k$ over $RG$ and a matrix $E$ over $\widehat{RG}_\xi$ with $\|E\|_\xi < 1$ such that $A = E_1 \cdots E_k(I - E)$, possibly after stabilizing $A$. Since $A$ and the $E_i$ are matrices over $RG$, we get that $E$ is also a matrix over $RG$. Now there is a small neighborhood of $\mathcal{U}$ of $[\xi]$ in $S(G)$ such that $\|E\|_\eta < 1$ for all $\eta$ with $[\eta] \in \mathcal{U}$. In particular $i_*\tau(A) = 0 \in K_1(\widehat{RG}_\eta)/\overline{W}_\eta$.

Since we also have $i_*\tau(A) = 0 \in K_1(\widehat{RG}_{-\xi})/\overline{W}_{-\xi}$, there is a small neighborhood $\mathcal{V}$ of $[-\xi]$ with $i_*\tau(A) = 0 \in K_1(\widehat{RG}_{-\eta})/\overline{W}_{-\eta}$ for all $\eta$ with $[-\eta] \in \mathcal{V}$. Since

$-\mathcal{V}$ is a neighborhood of $[\xi]$ we can find a rational $\eta$ with $[\eta] \in \mathcal{U} \cap -\mathcal{V}$ so that $\Delta(\tau_0) = 0 \in K_1(\widehat{RG}_\eta)/\overline{W}_\eta \oplus K_1(\widehat{RG}_{-\eta})/\overline{W}_{-\eta}$. But since $\eta$ is rational we get $\tau_0 = 0$. $\square$

**Corollary 7.2.** *Let $G$ be a group and $\xi : G \to \mathbb{R}$ a nonzero homomorphism. Then* $\mathrm{Wh}(G;\xi) = 0$ *if and only if* $\mathrm{Wh}(G) = 0$.

*Proof.* Observe that $g \to g^{-1}$ induces a ring isomorphism of $\widehat{\mathbb{Z}G}_\xi$ to the opposite ring of $\widehat{\mathbb{Z}G}_{-\xi}$. This induces an isomorphism $\mathrm{Wh}(G;\xi) \cong \mathrm{Wh}(G;-\xi)$ and the corollary follows from Corollary 3.4 and Theorem 7.1. $\square$

A natural question is whether the direct sum decomposition of (11) has a generalization to $K_1(\widehat{RG}_\xi)$, in particular one can ask if $\overline{W}_\xi$ is a direct summand. It may be possible to carry over the techniques of Pajitnov and Ranicki [20] at least for the ring $\Lambda_0$ of Section 5.

A similar question is whether we always have $W_\xi = \overline{W}_\xi$ as in the rational case. This would allow us to get a better understanding of $\overline{W}_\xi$ since Sheiham [27, Thm.B] gives a detailed description of $W_\xi$. To see this, note that the ring homomorphism $\varepsilon : \widehat{RG}^o_\xi \to RH$ given by restriction is a local augmentation in the sense of [27].

**The Latour obstruction.** Let $M$ be a closed connected smooth manifold with $\dim M \geq 6$ and denote $G = \pi_1(M)$. Then $\mathrm{Hom}(G, \mathbb{R}) = H^1(M;\mathbb{R})$ and such cohomology classes can be realized by closed 1-forms. Latour [12] gives two necessary and sufficient conditions for the existence of a nonsingular closed 1-form within a fixed cohomology class $\xi$. To describe the first homotopy theoretical condition let $X$ be a finite CW complex, $G = \pi_1(X)$, $\xi \in H^1(X;\mathbb{R})$ and $\tilde{X}$ the universal cover of $X$. Since $\mathbb{R}$ is contractible we can define a map $h : \tilde{X} \to \mathbb{R}$ such that

$$h(gx) \quad = \quad h(x) + \xi(g) \tag{12}$$

for all $x \in \tilde{X}$ and $g \in G$. Note that we regard $\xi$ as a homomorphism $\xi : G \to \mathbb{R}$ here. A map $h : \tilde{X} \to \mathbb{R}$ satisfying (12) is called a *height function for $\xi$*.

**Definition 7.3.** Let $X$ be a finite CW complex, $G = \pi_1(X)$ and $\xi \in H^1(X;\mathbb{R})$. Then $X$ is called *$\xi$-contractible*, if there exists a $G$-equivariant homotopy $H : \tilde{X} \times I \to \tilde{X}$ with $H_0 = \mathrm{id}_{\tilde{X}}$ and

$$h(H_1(x)) - h(x) \ \leq \ -\varepsilon \qquad \text{for all } x \in \tilde{X}$$

for some $\varepsilon > 0$ and height function $h : \tilde{X} \to \mathbb{R}$.

It is easy to see that $\xi$-contractibility does not depend on the height function or the $\varepsilon > 0$. Furthermore it is a homotopy invariant. For several equivalent conditions for $\xi$-contractibility we refer the reader to Latour [12, Prop.1.4]. By [12, Prop.1.10] $\xi$-contractibility implies that the completed cellular chain complex $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*(X)$ is acyclic. In that case we define

$$\tau_L(X, \xi) \quad = \quad \tau(\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*(X)) \in \mathrm{Wh}(G;\xi)$$

Latour's theorem then reads

**Theorem 7.4.** [12] *Let $M$ be a closed connected smooth manifold with* $\dim M \geq 6$ *and $\xi \in H^1(M;\mathbb{R})$. Then there exists a nonsingular closed 1-form $\omega$ representing $\xi$ if and only if $M$ is $(\pm\xi)$-contractible and $\tau_L(M, \xi) = 0 \in \mathrm{Wh}(G;\xi)$.*

In the case of an integer valued cohomology class $\xi \in H^1(M;\mathbb{Z}) = [M, S^1]$ the existence of a nonsingular closed 1-form representing $\xi$ is equivalent to the existence of a fibre bundle map $f : M \to S^1$ whose homotopy class represents $\xi$. This question was solved by Farrell [8, 9] and Siebenmann [28] who obtain an obstruction in $\mathrm{Wh}(G)$. An exposition of this case is given in Ranicki [21, §15], who also shows that the Farrell-Siebenmann obstruction is mapped to Latour's obstruction under the natural map $i_*$, see also [26].

Because of Corollary 3.4 we know in general that there is an element of $\mathrm{Wh}(G)$ that gets mapped to the Latour obstruction, but the question remains whether there is a natural geometric way to define an obstruction in $\mathrm{Wh}(G)$ that gets mapped to the Latour obstruction under $i_*$ as in the rational case. A partial answer to this is given in [26]. Let $\rho : \bar{M} \to M$ be the regular covering space corresponding to $\ker \xi$. By [26, Thm.1.3] we have that $\bar{M}$ is finitely dominated if and only if $M$ is $\eta$-contractible for every nonzero homomorphism $\eta : \pi_1(M) \to \mathbb{R}$ with $\ker \xi \subset \ker \eta$. In particular all Latour obstructions $\tau_L(M, \eta)$ are defined. Furthermore it is shown in [26] that all Farrell-Siebenmann obstructions for such rational $\eta$ agree and can be used as an obstruction for $\xi$. Note that $\bar{M}$ being finitely dominated is not necessary for $M$ to be $(\pm\xi)$-contractible if $\xi$ is not rational. Nevertheless we get the following corollary of Theorem 7.4.

**Corollary 7.5.** *Let $M$ be a closed connected smooth manifold with $\dim M \geq 6$ such that $\mathrm{Wh}(\pi_1(M)) = 0$ and let $\xi \in H^1(M;\mathbb{R})$. Then there exists a nonsingular closed 1-form $\omega$ representing $\xi$ if and only if $M$ is $(\pm\xi)$-contractible.* □

Whitehead groups can be very complicated but it is conjectured for example that $\mathrm{Wh}(\pi_1(M)) = 0$ for aspherical manifolds $M$. This conjecture has been verified in many special cases, in particular if $M$ is a compact manifold which admits a Riemannian metric of nonpositive sectional curvature, see Farrell and Jones [11]. For more examples of vanishing Whitehead groups of torsion-free groups see Lück and Reich [13, Thm.5.20.1] and the references given there.

**Localization.** In order to study the Morse theory of closed 1-forms, we can look at a subring of the Novikov ring $\widehat{\mathbb{Z}G}_\xi$ with $\xi : G \to \mathbb{R}$ injective using localization. For this let

$$S_\xi \quad = \quad \{1 - a \in \mathbb{Z}G \mid \|a\|_\xi < 1\},$$

a multiplicatively closed subset of $\mathbb{Z}G$. This gives rise to the inclusions of rings $\mathbb{Z}G \subset S_\xi^{-1}\mathbb{Z}G \subset \widehat{\mathbb{Z}G}_\xi$. This localization has some technical advantages over the Novikov ring. It appeared first in Farber [5] for the inclusion $\xi : \mathbb{Z} \to \mathbb{R}$ and more generally in Pajitnov [16].

In the case of an arbitrary homomorphism $\xi : G \to \mathbb{R}$ we can use a noncommutative localization in the sense of Cohn [2]. For this let $M(\mathbb{Z}G)$ be the set of all (finite) diagonal matrices over $\mathbb{Z}G$ and

$$\Sigma_\xi \quad = \quad \{I - A \in M(\mathbb{Z}G) \mid \|A\|_\xi < 1\}.$$

Then there exists a ring $\Sigma_\xi^{-1}\mathbb{Z}G$ together with a ring homomorphism $\varepsilon : \mathbb{Z}G \to \Sigma_\xi^{-1}\mathbb{Z}G$ such that $\varepsilon(M)$ is invertible for every $M \in \Sigma_\xi$ having the following universal property: For every ring $R$ and ring homomorphism $\rho : \mathbb{Z}G \to R$ such that $\rho(M)$

is invertible for every $M \in \Sigma_\xi$, there exists a unique ring homomorphism $\rho_1 : \Sigma_\xi^{-1}\mathbb{Z}G \to R$ such that $\rho = \rho_1\varepsilon$.

In particular the inclusion $\mathbb{Z}G \subset \widehat{\mathbb{Z}G}_\xi$ factors as $\mathbb{Z}G \to \Sigma_\xi^{-1}\mathbb{Z}G \to \widehat{\mathbb{Z}G}_\xi$.

This ring was first introduced in Pajitnov [19] and was also used by Farber and Ranicki [7] and Farber [6]. The main theorem of these papers can be stated as

**Theorem 7.6.** *Let $M$ be a closed smooth manifold with $G = \pi_1(M)$ and let $\xi \in H^1(M;\mathbb{R})$. Then for any closed 1-form $\omega$ having only Morse zeros and representing $\xi$ there exists a free chain complex $C_*^\omega$ over $\Sigma_\xi^{-1}\mathbb{Z}G$ such that $C_*^\omega$ is chain homotopy equivalent to the localized chain complex $\Sigma_\xi^{-1}\mathbb{Z}G \otimes_{\mathbb{Z}G} C_*(\tilde{M})$ and each $\Sigma_\xi^{-1}\mathbb{Z}G$-module $C_j^\omega$ has a canonical free basis which is in a one-to-one correspondence with the zeros of the closed 1-form $\omega$ of index $j$.*

To discuss the torsion of this equivalence, let

$$\mathrm{Wh}(G;\Sigma_\xi) \quad = \quad K_1(\Sigma^{-1}\mathbb{Z}G)/\langle \tau(\pm g), \tau(I-A) \,|\, g \in G, I - A \in \Sigma_\xi\rangle.$$

Clearly we get a factorization

$$\mathrm{Wh}(G) \longrightarrow \mathrm{Wh}(G;\Sigma_\xi) \longrightarrow \mathrm{Wh}(G;\xi).$$

Furthermore, if we denote the chain homotopy equivalence described in Theorem 7.6 by $\varphi : C_*^\omega \to \Sigma_\xi^{-1}\mathbb{Z}G \otimes_{\mathbb{Z}G} C_*(\tilde{M})$, we get $\tau(\varphi) = 0 \in \mathrm{Wh}(G;\Sigma_\xi)$. For rational $\xi$ this is shown in Ranicki [22], and the techniques of [22, §1] can be used to show that the chain collapse of [6] has zero torsion in $\mathrm{Wh}(G;\Sigma_\xi)$.

**Proposition 7.7.** *The natural map $i_* : \mathrm{Wh}(G;\Sigma_\xi) \to \mathrm{Wh}(G;\xi)$ is an isomorphism.*

*Proof.* It is surjective by Corollary 3.4, but note that we only need the proof of Lemma 5.1 to show surjectivity.

Let $A$ be an invertible matrix over $\Sigma_\xi^{-1}\mathbb{Z}G$. By Schofield [23, Thm.4.3] there exist matrices $B$ and $B'$ over $\mathbb{Z}G$ and a matrix $A'$ over $\Sigma_\xi^{-1}\mathbb{Z}G$ such that

$$B\begin{pmatrix} I & A' \\ 0 & A \end{pmatrix} = B' \quad \text{with} \quad B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ * & & B_n \end{pmatrix}$$

where each $B_i \in \Sigma_\xi$. In particular $B$ represents an invertible matrix over $\Sigma_\xi^{-1}\mathbb{Z}G$ with $\tau(B) = 0 \in \mathrm{Wh}(G;\Sigma_\xi)$. Therefore $B'$ is also invertible and $\tau(A) = \tau(B') \in \mathrm{Wh}(G;\Sigma_\xi)$.

Now if $i_*\tau(B') = 0 \in \mathrm{Wh}(G;\xi)$, then by Lemma 5.3 there exist elementary matrices $E_1, \ldots, E_k$ over $\mathbb{Z}G$ and a matrix $E$ necessarily over $\mathbb{Z}G$ with $\|E\|_\xi < 1$ and $B' = E_1 \cdots E_k(I - E)$. Note that $I - E \in \Sigma_\xi$, so $\tau(B') = 0 \in \mathrm{Wh}(G;\Sigma_\xi)$. □

## References

[1] M.M. Cohen, *A course in simple-homotopy theory*, Springer-Verlag, New York, 1973.

[2] P.M. Cohn, *Free rings and their relations*, London Mathematical Society Monographs, No. 2, Academic Press, London-New York, 1971.

[3] O. Cornea and A. Ranicki, *Rigidity and glueing for Morse and Novikov complexes*, J. Eur. Math. Soc. 5 (2003), 343-394.

[4] M. Damian, *Formes fermées non singulières et propriétés de finitude des groupes*, Ann. Sci. École Norm. Sup. (4) 33 (2000), 301-320.
[5] M. Farber, *Sharpness of the Novikov inequalities*, Functional Anal. Appl. 19 (1985), 40-48.
[6] M. Farber, *Morse-Novikov critical point theory, Cohn localization and Dirichlet units*, Commun. Contemp. Math. 1 (1999), 467-495.
[7] M. Farber and A. Ranicki, *The Morse-Novikov theory of circle-valued functions and noncommutative localization*, Proc. Steklov Inst. Math. 225 (1999), 363-371.
[8] F.T. Farrell, *The obstruction to fibering a manifold over a circle*, Yale University Ph.D. thesis, 1967, published in Indiana Univ. Math. J. 21 (1971), 315-346.
[9] F.T. Farrell, *The obstruction to fibering a manifold over a circle*. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, 69-72, Gauthier-Villars, 1971.
[10] F.T. Farrell and W.C. Hsiang, *A formula for* $K_1 R_\alpha[T]$. 1970 Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968) pp. 192-218.
[11] F.T. Farrell and L.E. Jones, *Stable pseudoisotopy spaces of compact non-positively curved manifolds*, J. Differential Geom. 34 (1991), 769-834.
[12] F. Latour, *Existence de 1-formes fermées non singulières dans une classe de cohomologie de de Rham*, Publ. IHES 80 (1994), 135-194.
[13] W. Lück and H. Reich, *The Baum-Connes and the Farrell-Jones conjecture in* $K$*- and* $L$*-theory*, Handbook of $K$-theory Volume 2, editors: E.M. Friedlander, D.R. Grayson, 703-842, Springer Verlag, 2005.
[14] S.P. Novikov, *Multivalued functions and functionals. An analogue of Morse theory*, Soviet Math. Dokl. 24 (1981), 222-226.
[15] S.P. Novikov, *The Hamiltonian formalism and a multivalued analogue of Morse theory*, Russian Math. Surveys 37:5 (1982), 1-56.
[16] A. Pazhitnov, *On the sharpness of Novikov type inequalities for manifolds with free abelian fundamental group*, Math. USSR-Sb. 68 (1991) 351-389.
[17] A. Pazhitnov, *On the Novikov complex for rational Morse forms*, Ann. Fac. Sci. Toulouse 4 (1995), 297-338.
[18] A. Pazhitnov, *Surgery on the Novikov complex*, K-theory 10 (1996), 323-412.
[19] A. Pajitnov, *Rationality and exponential growth properties of the boundary operators in the Novikov complex*, Math. Res. Lett. 3 (1996), 541-548.
[20] A. Pajitnov and A. Ranicki, *The Whitehead group of the Novikov ring*, K-Theory 21 (2000), 325-365.
[21] A. Ranicki, *High-dimensional knot theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 1998.
[22] A. Ranicki, *The algebraic construction of the Novikov complex of a circle-valued Morse function*, Math. Ann. 322 (2002), 745-785.
[23] A. Schofield, *Representation of rings over skew fields*, London Mathematical Society Lecture Note Series, 92, Cambridge University Press, Cambridge, 1985.
[24] D. Schütz, *Controlled connectivity of closed 1-forms*, Algebr. Geom. Topol. 2 (2002), 171-217.
[25] D. Schütz, *Geometric chain homotopy equivalences between Novikov complexes*, High-dimensional manifold topology, 469-498, World Sci. Publishing, River Edge, NJ, 2003.
[26] D. Schütz, *Finite domination, Novikov homology and nonsingular closed 1-forms*, to appear in Math. Z.
[27] D. Sheiham, *Whitehead groups of localizations and the endomorphism class group*, J. Algebra 270 (2003), 261-280.
[28] L. Siebenmann, *A total Whitehead torsion obstruction to fibering over the circle*, Comment. Math. Helv. 45 (1970), 1-48.
[29] J.-C. Sikorav, *Points fixes de difféomorphismes symplectiques, intersectiones de sous-variétés lagrangiennes, et singularités de un-formes fermées*, Thése, Université Paris-Sud, Centre d'Orsay, 1987.

Department of Mathematical Sciences, University of Durham, Science Laboratories, South Rd, Durham DH1 3LE, United Kingdom

*E-mail address*: dirk.schuetz@durham.ac.uk