NOVIKOV-BETTI NUMBERS AND THE FUNDAMENTAL GROUP

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Theorem 1. *Let X be a connected finite polyhedron and let $\xi \in H^1(X;\mathbf{R})$ be a nonzero cohomology class. If the first Novikov-Betti number $b_1(\xi)$ is nonzero, $b_1(\xi) > 0$, then $\pi_1(X)$ contains a nonabelian free subgroup.*

This result may appear striking as the Novikov-Betti numbers carry "abelian" information about X. We refer to [4], [3] for the definition of the Novikov-Betti numbers; an explicit definition will also be given below in the proof of Theorem 1.

An alternative description of $b_i(\xi)$ uses homology of complex flat line bundles. Consider the variety $\mathcal{V}_\xi$ of all complex flat line bundles L over X having the following property: L has trivial monodromy along any loop γ in X assuming that $\langle \xi, [\gamma] \rangle = 0$. It is easy to see that (a) $\mathcal{V}_\xi$ is an algebraic variety isomorphic to $(\mathbf{C}^*)^r$ for some integer r and (b) the dimension $\dim H_i(X;L)$ is independent of L assuming that $L \in \mathcal{V}_\xi$ is generic, see [3], Theorem 1.50. The number $\dim H_i(X;L)$ for a generic $L \in \mathcal{V}_\xi$ coincides with the Novikov-Betti number $b_i(\xi)$.

The proof of Theorem 1 given below is based on the results of R. Bieri, W. Neumann, R. Strebel [1] and J.-Cl. Sikorav [6].

Theorem 1 implies the following vanishing result:

Corollary 1. *Let X be a connected finite polyhedron having an amenable fundamental group. Then the first Novikov-Betti number vanishes $b_1(\xi) = 0$ for any $\xi \neq 0 \in H^1(X;\mathbf{R})$.*

Corollary 1 follows from Theorem 1 as an amenable group contains no nonabelian free subgroups. As another simple corollary of Theorem 1 we mention the next statement:

Corollary 2. *Assume that X is a connected finite two-dimensional polyhedron. If the Euler characteristic of X is negative $\chi(X) < 0$ then $\pi_1(X)$ contains a nonabelian free subgroup.*

Proof. The first Betti number $b_1(X)$ is positive as follows from $\chi(X) < 0$. Hence there exists a nonzero cohomology class $\xi \in H^1(X;\mathbf{R})$. Then

$$\chi(X) = b_2(\xi) - b_1(\xi), \tag{1}$$

see Proposition 1.40 in [3]. Here we use that $b_0(\xi) = 0$ for $\xi \neq 0$, see Corollary 1.33 in [3]. The inequality $\chi(X) < 0$ together with (1) imply that $b_1(\xi) > 0$. Theorem 1 now states that $\pi_1(X)$ contains a nonabelian free subgroup. □

Corollary 2 is known, it was obtain by N.S. Romanovskii, see [5]. It can be equivalently expressed algebraically as follows:

Corollary 3. *Any discrete group G of deficiency greater than 1 contains a nonabelian free subgroup.*

The authors thank Jonathan Hillman for referring to [5].

Proof of Theorem 1. Consider the following diagram of rings and ring homomorphisms:

$$\begin{array}{ccc} & \Lambda & \\ \alpha \swarrow & & \searrow \rho \\ S_\xi & \xrightarrow{\beta} & N_\xi \end{array} \tag{2}$$

Here $\Lambda = \mathbf{Z}[\pi]$ is the group ring of the fundamental group of X which we denote $\pi = \pi_1(X, x_0)$.

The ring N_ξ is the *Novikov ring* which is defined as follows. View the class ξ as a group homomorphism $\xi : \pi \to \mathbf{Z}$ and let H denotes the factorgroup $\pi/\mathrm{Ker}(\xi)$. It is a finitely generated free abelian group. The usual group ring $\mathbf{Z}[H]$ consists of finite sums of the form $\sum a_j h_j$ with $a_j \in \mathbf{Z}$ and $h_j \in H$; it coincides with the Laurent polynomial ring in $\mathrm{rk} H$ variables. The Novikov ring N_ξ is a completion of $\mathbf{Z}[H]$; its elements are infinite sums $\sum_{j=1}^{\infty} a_j h_j$ with $a_j \in \mathbf{Z}$ and $h_j \in H$ such that the sequence of evaluation $\xi(h_j)$ tends to $-\infty$. In other words, N_ξ is the ring of Laurent power series in $\mathrm{rk} H$ variables where the terms of the series go to infinity in the direction specified by the class ξ.

The ring S_ξ is the *Novikov - Sikorav* completion of the group ring $\Lambda = \mathbf{Z}[\pi]$; it was originally introduced by J.-Cl. Sikorav [6]. Elements of S_ξ are infinite sums of the form $\sum_{j=1}^{\infty} a_j g_j$ where $a_j \in \mathbf{Z}$, $g_j \in \pi$ and $\xi(g_j) \to -\infty$.

Diagram (2) includes the obvious rings homomorphisms α, β, ρ and is commutative.

Let X be a finite connected polyhedron with fundamental group π. Consider the universal covering $\tilde{X} \to X$. The cellular chain complex $C_*(\tilde{X})$ is a complex of finitely generated free left Λ-modules. Since N_ξ is a commutative ring and a principal ideal domain (see [3], Lemma 1.10) the homology $H_i(N_\xi \otimes_\Lambda C_*(\tilde{X}))$ is a finitely generated N_ξ-module and its rank (over N_ξ) equals the Novikov-Betti number $b_i(\xi)$, see [3], §1.5.

Our first goal is to show that the assumption $b_1(\xi) \neq 0$ implies

$$H_1(S_\xi \otimes_\Lambda C_*(\tilde{X})) \neq 0. \tag{3}$$

Indeed, note that $H_0(S_\xi \otimes_\Lambda C_*(\tilde{X})) = 0$ (as we assume that $\xi \neq 0$). Hence, using the Künneth spectral sequence we find that

$$\begin{aligned} H_1(N_\xi \otimes_\Lambda C_*(\tilde{X})) &= H_1(N_\xi \otimes_{S_\xi} (S_\xi \otimes_\Lambda C_*(\tilde{X}))) \\ &= N_\xi \otimes_{S_\xi} H_1(S_\xi \otimes_\Lambda C_*(\tilde{X})). \end{aligned} \tag{4}$$

Hence $b_1(\xi) \neq 0$ implies the nonvanishing of $H_1(N_\xi \otimes_\Lambda C_*(\tilde{X}))$ which via (4) gives (3).

Note that $b_1(\xi) = b_1(-\xi)$ (see Corollary 1.31 in [3]) and hence the nonvanishing of the Novikov-Betti number $b_1(\xi)$ implies the nonvanishing of the Novikov - Sikorav homology $H_1(S_\xi \otimes_\Lambda C_*(\tilde{X}))$ for both cohomology classes ξ and $-\xi$.

R. Bieri, W. Neumann and R. Strebel [1] defined a geometric invariant of discrete groups. It can be viewed as a subset Σ of the space of nonzero cohomology classes $\Sigma \subset H^1(X; \mathbf{R}) - \{0\}$. A theorem of Jean-Claude Sikorav (see [6], page 86 or [2]) states that for a nonzero cohomology class $\xi \in H^1(X; \mathbf{R})$ the following conditions are equivalent: (a) $H_1(S_\xi \otimes_\Lambda C_*(\tilde{X})) = 0$ and (b) $-\xi \in \Sigma$. Hence, as explained above, $b_1(\xi) \neq 0$ implies that $\xi \notin \Sigma$ and $-\xi \notin \Sigma$. Now we apply Theorem C from [1] which states that the union of Σ and $-\Sigma$ equals $H^1(X;\mathbf{R}) - \{0\}$ assuming that $\pi_1(X)$ has no non-abelian free subgroups. Since in our case neither $\xi \in \Sigma$ nor $-\xi \in \Sigma$ we conclude that $\pi_1(X)$ must contain a non-abelian free subgroup. □

Next we mention an example showing that a space with nonvanishing Novikov torsion number $q_1(\xi)$ may have an amenable fundamental group. In other words, Theorem 1 becomes false if one replaces the assumption $b_1(\xi) \neq 0$ by the assumption $q_1(\xi) \neq 0$. Example 1.49 in [3] gives a two-dimensional polyhedron with $b_1(\xi) = 0$ and $q_1(\xi) = 1$ for some nonzero $\xi \in H^1(X;\mathbf{R})$. The fundamental group of X is a Baumslag-Solitar group $G = |a, b; aba^{-1} = b^2|$. The later group appears in the exact sequence $0 \to \mathbf{Z}[\frac{1}{2}] \to G \to \mathbf{Z} \to 0$ (see [3], page 30); hence G is amenable.

References

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