FINITE DOMINATION, NOVIKOV HOMOLOGY AND NONSINGULAR CLOSED 1-FORMS

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Abstract. Let X be a finite connected CW-complex and $\rho : \overline{X} \to X$ a regular covering space with free abelian covering transformation group. For $\xi \in H^1(X;\mathbb{R})$ we define the notion of ξ-contractibility of X. This notion is closely related to the vanishing of the Novikov homology of the pair (X,ξ). We show that finite domination of $\overline{X}$ is equivalent to X being ξ-contractible for every nonzero ξ with $\rho^*\xi = 0 \in H^1(\overline{X};\mathbb{R})$. If M is a closed connected smooth manifold the condition that M is ξ-contractible is necessary for the existence of a nonsingular closed 1-form representing ξ. Also ξ-contractibility guarantees the definition of the Latour obstruction $\tau_L(M,\xi)$ whose vanishing is then sufficient for the existence of a nonsingular closed 1-form provided $\dim M \geq 6$. Now if $\rho : \overline{M} \to M$ is a finitely dominated regular $\mathbb{Z}^k$-covering space we get that $\tau_L(M,\xi)$ is defined for every nonzero ξ with $\rho^*\xi = 0$ and the vanishing of one such obstruction implies the vanishing of all such $\tau_L(M,\xi)$.

1. Introduction

Given an element $\xi \in H^1(M;\mathbb{R})$ where M is a closed connected smooth manifold, it can be represented by a closed 1-form on M. Provided that $\dim M \geq 6$ Latour [8] gave necessary and sufficient conditions for the existence of a nonsingular closed 1-form representing ξ. In the case that ξ is actually an integer valued cohomology class the existence of a nonsingular closed 1-form representing ξ is equivalent to the existence of a smooth fibre bundle map $f : M \to S^1$ which represents ξ. Necessary and sufficient conditions for the existence of such a smooth fibre bundle map were already given by Farrell [4, 5], see also Siebenmann [17].

Theorem 1.1 (Farrell [5]). *Let M be a closed connected smooth manifold with $\dim M \geq 6$ and let $f : M \to S^1$ represent a nonzero cohomology class in $H^1(M;\mathbb{Z}) \cong [M,S^1]$. Then f is homotopic to a smooth fibre bundle map if and only if $\overline{M}$ is finitely dominated and $\tau_F(M,f) = 0 \in \mathrm{Wh}(\pi_1(M))$.*

Here $\overline{M}$ is the infinite cyclic covering space corresponding to $\ker(f_\# : \pi_1(M) \to \pi_1(S^1))$ and $\tau_F(M,f)$ is a naturally defined obstruction whose definition is given in Section 5. On the other hand, Latour's theorem is given by

Theorem 1.2 (Latour [8]). *Let M be a closed connected smooth manifold with $\dim M \geq 6$ and let $\xi \in H^1(M;\mathbb{R})$ be nonzero. Then ξ can be represented by a nonsingular closed 1-form if and only if M is $(\pm\xi)$-contractible and $\tau_L(M,\xi) = 0 \in \mathrm{Wh}(\pi_1(M);\xi)$.*

The condition that M is $(\pm\xi)$-contractible is given in Definition 2.2. Latour actually uses a different, but equivalent condition, see Remark 2.3 for details. Again $\tau_L(M,\xi)$ is a naturally defined obstruction, but in a group which depends on ξ. Indeed the group $\mathrm{Wh}(\pi_1(M);\xi)$ is an algebraic K-group of a completion of the group ring $\mathbb{Z}\pi_1(M)$ and there is a natural homomorphism $i_* : \mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}(\pi_1(M);\xi)$.

Ranicki [13] has shown that for $\xi \in H^1(M;\mathbb{Z})$ finite domination of the infinite cyclic covering space $\overline{M}$ is equivalent to M being $(\pm\xi)$-contractible. Furthermore he showed in [14, §15] that $i_*\tau_F(M,f) = \tau_L(M,f_\#)$ so Theorem 1.2 is indeed a generalization of Theorem 1.1. But for $\xi \in H^1(M;\mathbb{R})$ which do not come from circle valued maps there is the problem that the groups $\mathrm{Wh}(\pi_1(M);\xi)$ are not very well understood. It is also not clear whether an obstruction can be defined in $\mathrm{Wh}(\pi_1(M))$. One aim of this paper is to give conditions under which the obstruction $\tau_L(M,\xi)$ actually does come from $\mathrm{Wh}(\pi_1(M))$.

Notice that $H^1(M;\mathbb{R}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{R})$ so we can think of the cohomology class ξ as a homomorphism $\xi : \pi_1(M) \to \mathbb{R}$. Denote $\overline{M}$ the covering space of M corresponding to $\ker\xi$. It is easy to see that there is a nonnegative integer k such that $\overline{M} \to M$ is a $\mathbb{Z}^k$-covering. We already remarked that in the case $k = 1$ M being $(\pm\xi)$-contractible is equivalent to $\overline{M}$ being finitely dominated. In general it is easy to give examples where M is $(\pm\xi)$-contractible, but $\overline{M}$ is not finitely dominated. On the other hand we do get that M is $(\pm\xi)$-contractible if $\overline{M}$ is finitely dominated, but we also get ξ'-contractibility for many other homomorphisms. More precisely we get the following result.

Theorem 1.3. *Let X be a finite connected CW-complex and $N \le \pi_1(X)$ a normal subgroup such that $\pi_1(X)/N \cong \mathbb{Z}^k$ for some $k \ge 0$. Denote $\overline{X}$ the regular covering space corresponding to N. Then $\overline{X}$ is finitely dominated if and only if X is ξ-contractible for all nonzero homomorphisms $\xi : \pi_1(X) \to \mathbb{R}$ with $N \le \ker\xi$.*

The case $k = 1$ is proven in Ranicki [13]. If X is aspherical it also follows from the work of Bieri and Renz [1, §5] in which case finite domination can be replaced by homotopy finite. In general finite domination cannot be replaced by homotopy finite, compare Example 5.11.

The proof that finite domination implies ξ-contractibility is an induction argument based on the proof in Ranicki [13]. The other direction is more complicated and is based on arguments used by Bieri and Renz [1, §5]. In the case $k = 1$ Ranicki [13] has a more elegant argument but we do not know how to generalize it.

An unpublished result of Farrell states that if two maps $f, g : M \to S^1$ represent linearly independent elements of $H^1(M;\mathbb{Z})$ and the $\mathbb{Z}^2$-covering space $\overline{M}$ corresponding to $\ker f_\# \cap \ker g_\#$ is finitely dominated, then $\tau_F(M,f) = \tau_F(M,g)$. So f is homotopic to a fibre bundle map if and only if the same holds for g. Because of Theorem 1.3 we know that finite domination of $\overline{M}$ is equivalent to the ξ-contractibility of M for every nonzero $\xi : \pi_1(M) \to \mathbb{R}$ which vanishes on $\ker f_\# \cap \ker g_\#$. Thus we expect an impact on the obstructions $\tau_L(M,\xi)$ for such ξ as well. Indeed it turns out that in this situation $\tau_F(M,f)$ determines every such $\tau_L(M,\xi)$ via $i_* : \mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}(\pi_1(M);\xi)$. Combining this with Theorem 1.2 we get

Theorem 1.4. *Let M be a closed connected smooth manifold with $\dim M \ge 6$, $N \le \pi_1(M)$ a normal subgroup such that $\pi_1(M)/N \cong \mathbb{Z}^k$ for some $k \ge 1$ and such that the covering space corresponding to N is finitely dominated. Then the following are equivalent.*

(1) *There is a nonzero* $\xi : \pi_1(M) \to \mathbb{R}$ *with* $N \leq \ker \xi$ *which can be represented by a nonsingular closed 1-form.*
(2) *Every nonzero* $\xi : \pi_1(M) \to \mathbb{R}$ *with* $N \leq \ker \xi$ *can be represented by a nonsingular closed 1-form.*

So in the situation of Theorem 1.4 we either get that all nonzero homomorphisms vanishing on N can be represented by a nonsingular closed 1-form or none of them.

I would like to thank Thomas Farrell, Ross Geoghegan and Andrew Ranicki for valuable discussions. This research was supported by the SFB 478 “Geometrische Strukturen in der Mathematik” at the University Münster.

2. Basic definitions

Definition 2.1. A topological space X is called *finitely dominated* if there exists a finite CW-complex K and maps $a : K \to X$, $b : X \to K$ such that $ab \simeq \mathrm{id}_X : X \to X$. The space X is called *homotopy finite* if it is homotopy equivalent to a finite CW-complex.

Obviously a homotopy finite space is finitely dominated. Conversely for a finitely dominated space X Wall [19] defined a finiteness obstruction $[X] \in \tilde{K}_0(\mathbb{Z}\pi_1(X))$ such that X is homotopy finite if and only if $[X] = 0$.

We will be interested in the finite domination properties of $\mathbb{Z}^k$-covers of a finite CW-complex X for integers $k \geq 1$. The case $k = 1$ was studied by Ranicki [13]. To do this we will study homomorphisms $\xi : \pi_1(X) \to \mathbb{R}$ which give rise in a natural way to $\mathbb{Z}^k$-covering spaces of X.

Let X be a connected finite CW-complex. We denote the universal covering space of X by $\tilde{X}$. Since $H^1(X; \mathbb{R}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{R})$ we think of elements $\xi \in H^1(X; \mathbb{R})$ as homomorphisms $\xi : \pi_1(X) \to \mathbb{R}$. We do not have to worry about basepoints as $\mathbb{R}$ is commutative. Now given a homomorphism $\xi : \pi_1(X) \to \mathbb{R}$ there is a nonnegative integer k such that $\pi_1(X)/\ker \xi \cong \mathbb{Z}^k$ since X is a finite complex. More generally if $N \leq \pi_1(X)$ is a normal subgroup, we denote $X_N = \tilde{X}/N$, a regular covering space of X. In particular for $N = \ker \xi$ we get that X_N is a $\mathbb{Z}^k$-covering of X. Given a covering space $\rho : \overline{X} \to X$, we denote the group of covering transformations by $\Delta(\overline{X} : X)$.

Notice that $\pi_1(X)$ acts on $\tilde{X}$ by covering transformations and on $\mathbb{R}$ by $g \cdot x = x + \xi(g)$ for $g \in \pi_1(X)$ and $x \in \mathbb{R}$. Since $\mathbb{R}$ is contractible we can find an equivariant map $h : \tilde{X} \to \mathbb{R}$, that is a map with $h(gx) = h(x) + \xi(g)$. Such an equivariant map h will be called a *control function* for ξ.

Definition 2.2. Let $\varepsilon > 0$ and $h : \tilde{X} \to \mathbb{R}$ a control function for ξ. We say that X is *ξ-contractible* if there is an equivariant homotopy $H : \tilde{X} \times [0, 1] \to \tilde{X}$ with $H_0 = \mathrm{id}_{\tilde{X}}$ and

$$hH_1(x) - h(x) \quad \leq \quad -\varepsilon$$

for all $x \in \tilde{X}$.

It is easy to see that this definition does not depend on ε or h. In fact we can easily increase the $\varepsilon > 0$ by iterating the homotopy. Then the condition does not depend on h because X is a finite complex. Also the control function h factors through X_N with $N = \ker \xi$ so the condition of X being ξ-contractible can also be tested using X_N. We will write X is $(\pm\xi)$-contractible if X is both ξ-contractible and $(-\xi)$-contractible.

Remark 2.3. The condition that X is ξ-contractible is equivalent to other conditions which have appeared in the literature. In [8, §1] Latour defines the space $\mathcal{C}_\xi(X)$ to be the set of maps $\gamma : [0,\infty) \to X$ with the property that they lift to a map $\tilde{\gamma} : [0,\infty) \to \tilde{X}$ such that $\lim_{t\to\infty} h\tilde{\gamma}(t) = -\infty$. Equipped with an appropriate topology the evaluation map $e : \mathcal{C}_\xi(X) \to X$ defined by $e(\gamma) = \gamma(0)$ is a fibration. Let $\mathcal{M}_\xi(X)$ be the fibre of e. It is easy to see that X is ξ-contractible if and only if there is a section $s : X \to \mathcal{C}_\xi(X)$ of e. Now Latour [8, Prop.1.4] shows that the existence of a section is equivalent to $e : \mathcal{C}_\xi(X) \to X$ being a homotopy equivalence and equivalent to $\mathcal{M}_\xi(X)$ being contractible.

Farber [3] defines a Lusternik-Schnirelmann category for the pair (X, ξ) where X is a finite CW-complex and $\xi \in H^1(X;\mathbb{R})$, denoted by $\mathrm{cat}(X,\xi)$. This is a nonnegative integer and for a connected X it follows directly from the definitions that $\mathrm{cat}(X,\xi) = 0$ if and only if X is ξ-contractible.

Thus we get from Farber [3, Lm.3.6] homotopy invariance.

Lemma 2.4. *Let X be a finite connected CW-complex, $\xi : \pi_1(X) \to \mathbb{R}$ a homomorphism such that X is ξ-contractible. Let Y be a finite connected CW-complex and $\varphi : Y \to X$ a homotopy equivalence. Then Y is $\varphi^*\xi$-contractible where $\varphi^*\xi = \xi \circ \varphi_\# : \pi_1(Y) \to \mathbb{R}$.* □

Another easy observation is that X being ξ-contractible implies that $\xi \neq 0$ and that X is $c\xi$-contractible for every $c > 0$.

We thus define

$$S(\pi_1(X)) \quad = \quad H^1(X;\mathbb{R}) - \{0\}/\sim$$

where $\xi_1, \xi_2 \in H^1(X;\mathbb{R}) - \{0\}$ are equivalent if there is a $c > 0$ such that $\xi_1 = c\xi_2$. Clearly $S(\pi_1(X))$ is an $(r-1)$-sphere, where r is the first Betti number of X. In particular we have a natural topology on $S(\pi_1(X))$.

By abuse of notation we will write $\xi \in S(\pi_1(X))$ for a nonzero homomorphism $\xi : \pi_1(X) \to \mathbb{R}$. We also define

$$\Sigma(X) \quad = \quad \{\xi \in S(\pi_1(X)) \,|\, X \text{ is } \xi\text{-contractible}\}$$

Remark 2.5. The notation $\Sigma(X)$ is motivated by Bieri and Renz [1]. In the case that X is aspherical $\Sigma(X)$ coincides with ${}^*\Sigma^m(\pi_1(X))$, where $m = \dim X$, as defined in [1, Rm.6.5].

We can refine the definition of $\Sigma(X)$ to $\Sigma^k(X)$ analogously to [1] by requiring the homotopy H in Definition 2.2 to be only defined on the k-skeleton of $\tilde{X}$. This leads to refinements to some of our results but at the moment we will stick to the absolute case.

We have the following openness result for $\Sigma(X)$.

Proposition 2.6. *Let X be a finite connected CW-complex. Then $\Sigma(X)$ is an open subset of $S(\pi_1(X))$.*

Proof. Notice that $S(\pi_1(X)) = \mathrm{Hom}(\pi_1(X),\mathbb{R}) - \{0\}/\sim$ and the topology is induced from the compact-open topology where $\pi_1(X)$ is considered discrete.

Let $r = b_1(X)$, the first Betti number of X. There is an epimorphism $\varepsilon : \pi_1(X) \to \mathbb{Z}^r$ and $\pi_1(X)$ acts on $\mathbb{R}^r$ by translation corresponding to ε. Let $h : \tilde{X} \to \mathbb{R}^r$ be an equivariant map and $S^{r-1} \subset \mathbb{R}^r$ the standard sphere. For every $\xi \in S(\pi_1(X))$ there is a unique $x_\xi \in S^{r-1}$ such that $h_\xi : \tilde{X} \to \mathbb{R}$ given by $h_\xi(x) = \langle x, x_\xi \rangle$ is a control function for ξ. Here $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

Now let $\xi \in \Sigma(X)$. There exists an $\varepsilon > 0$ and an equivariant homotopy $H : \tilde{X} \times [0,1] \to \tilde{X}$ such that $h_\xi H_1(x) - h_\xi(x) \leq -\varepsilon$ for all $x \in \tilde{X}$. Since H is an equivariant homotopy, we have that $|hH_1(x) - h(x)| \leq S$ for some $S > 0$, where $|\cdot|$ is the Euclidean norm. So for $\xi' \in S(\pi_1(X))$ we get

$$\begin{aligned} h_{\xi'} H_1(x) - h_{\xi'}(x) &= \langle hH_1(x) - h(x), x_{\xi'} \rangle \\ &= \langle hH_1(x) - h(x), x_\xi \rangle + \langle hH_1(x) - h(x), x_{\xi'} - x_\xi \rangle \\ &\leq -\varepsilon + S \cdot |x_{\xi'} - x_\xi| \end{aligned}$$

So for $|x_{\xi'} - x_\xi| < \frac{\varepsilon}{2S}$ we have that X is ξ'-contractible. This shows that $\xi' \in \Sigma(X)$ if ξ' is close enough to ξ. □

Prototypes for ξ-contractible spaces are given by mapping tori.

Definition 2.7. Let X be a topological space and $f : X \to X$ a map. Then the *mapping torus* $T_f = T(f : X \to X)$ is defined to be the quotient space $X \times [0,1]/\sim$ with $(x, 0) \sim (fx, 1)$.

If X is a finite CW-complex and f is cellular, then T_f has a natural structure as a finite CW-complex. Also there is an obvious map $g : T_f \to S^1$ given by $g([x,t]) = [t] \in S^1 = \mathbb{R}/\mathbb{Z}$.

Also T_f has a natural infinite cyclic covering space corresponding to g defined by

$$\overline{T}_f \quad = \quad \coprod_{n=-\infty}^{\infty} X \times [0,1] \times \{n\}/\sim$$

with $(x, 0, n) \sim (fx, 1, n-1)$. A covering transformation generating the covering transformation group is given by $[x, t, n] \mapsto [x, t, n+1]$ and there is a natural control map $h : \overline{T}_f \to \mathbb{R}$ given by $h([x, t, n]) = t + n$.

If $f, g : X \to X$ are homotopic then T_f and T_g are homotopy equivalent. Another useful property is the following proposition which goes back to Mather [9].

Proposition 2.8. *Let $f : X \to Y$ and $g : Y \to X$ be maps between topological spaces. Then $\varphi_f : T_{gf} \to T_{fg}$ and $\varphi_g : T_{fg} \to T_{gf}$ given by $\varphi_f([x,t]) = [fx, t]$ and $\varphi_g([y,t]) = [gy, t]$ are mutually inverse homotopy equivalences.*

Moreover, if X and Y are finite CW-complexes and f and g are cellular, these equivalences are simple.

Proof. See Hughes and Ranicki [7, Prop.14.2]. □

The natural projection $g : T_f \to S^1$ induces a surjective homomorphism $g_\# : \pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$.

Lemma 2.9. *If X is a finite connected CW-complex and $f : X \to X$ is cellular, then T_f is $g_\#$-contractible.*

Proof. It is enough to define a $\mathbb{Z}$-equivariant homotopy $H : \overline{T}_f \times [0,1] \to \overline{T}_f$ with the necessary properties. Let

$$H([x,t,n]) \quad = \quad \begin{cases} [x, t-s, n] & t-s \geq 0 \\ [fx, 1+t-s, n-1] & t-s \leq 0 \end{cases}$$

One easily checks that this is well defined and has the desired properties with respect to the natural control map. □

Farber [3, Ex.3.4] shows that for a map $f : S^2 \to S^2$ of degree 2 we get that T_f is $g_\#$-contractible, but not $(-g_\#)$-contractible. It would be interesting to have a closed manifold M with the property that M is ξ-contractible but not $(-\xi)$-contractible for some homomorphism $\xi : \pi_1(M) \to \mathbb{R}$.

The following lemma is well known, see Farrell [5] or Ranicki [13].

Lemma 2.10. *Let X be a connected CW-complex, $\rho : \overline{X} \to X$ an infinite cyclic covering space and $t : \overline{X} \to \overline{X}$ a generator of the covering transformation group. Then $\varphi : T(t : \overline{X} \to \overline{X}) \to X$ given by $\varphi([x,t]) = \rho(x)$ is a homotopy equivalence.*

Proof. Observe that $T(t : \overline{X} \to \overline{X}) = \overline{X} \times_{\mathbb{Z}} \mathbb{R}$ and φ is a fibration with fibre $\mathbb{R}$. □

We get the following useful corollary to Lemma 2.10, compare Mather [9].

Corollary 2.11. *Let X be a connected CW-complex and $\rho : \overline{X} \to X$ a $\mathbb{Z}^k$-covering space for some $k \geq 1$. Assume that $\overline{X}$ is finitely dominated. Then X is homotopy finite.*

Proof. The proof is by induction on k. Let $k = 1$, K a finite CW-complex and $a : K \to \overline{X}$, $b : \overline{X} \to K$ be maps with $ab \simeq \mathrm{id}_{\overline{X}}$. By Lemma 2.10 and Proposition 2.8 we have

$$X \simeq T(t : \overline{X} \to \overline{X}) \simeq T(tab : \overline{X} \to \overline{X}) \simeq T(bta : K \to K)$$

and the space on the right is a finite CW-complex.

If $k > 1$ we can factor $\rho : \overline{X} \to X$ into $\rho_1 : \overline{X} \to X_1$ and $\rho_2 : X_1 \to X$ where ρ_1 is a $\mathbb{Z}^{k-1}$-covering and ρ_2 is a $\mathbb{Z}$-covering. By induction we get that X_1 is homotopy finite, hence finitely dominated. Again by induction X is homotopy finite. □

We now want to examine how mapping tori behave with G-actions. So let G be a group and X, Y be G-spaces. Let $f : X \to X$ and $h : X \to Y$ be G-maps. The proof of the following lemma is straightforward.

Lemma 2.12. *In the above situation T_f is also a G-space with action $g \cdot [x,t] = [gx,t]$. Furthermore if h and $h \circ f$ are equivariantly homotopic, we get a G-equivariant map $\varphi : T_f \to Y$ by setting $\varphi([x,t]) = H(x,t)$, where $H : X \times [0,1] \to Y$ is an equivariant homotopy $H : h \circ f \simeq h$.* □

Notice that in this situation $\overline{T}_f$ is a $\mathbb{Z} \times G$-space with the obvious action.

We will also need equivariant versions of Proposition 2.8 and Lemma 2.10, the proofs extend to the equivariant setting.

Proposition 2.13. *Let X and Y be G-spaces, $f : X \to Y$ and $g : Y \to X$ G-equivariant maps. Then T_{gf} and T_{fg} are G-equivariant homotopy equivalent via the maps given in Proposition 2.8.* □

Lemma 2.14. *Let X be a finite connected CW-complex, $\rho : \overline{X} \to X$ a $G \times \mathbb{Z}$-covering space and $X_1 = \overline{X}/\mathbb{Z}$. Let $t : \overline{X} \to \overline{X}$ be a covering transformation generating $\Delta(\overline{X} : X_1)$. Then $\varphi : T(t : \overline{X} \to \overline{X}) \to X$ given by $\varphi([x, s]) = \rho_1(x)$ is a G-equivariant homotopy equivalence.* □

Finally we need the following lemma whose proof is easy to see.

Lemma 2.15. *Let $\rho : \overline{X} \to X$ be a G-covering and $h : X \to X$ a map which is covered by a G-equivariant map $\bar{h} : \overline{X} \to \overline{X}$. Then the natural map $T_{\bar{h}} \to T_h$ is a G-covering space.* □

3. Relations between finite domination and ξ-contractibility

Let N be a subgroup of $\pi_1(X)$ which contains the commutator subgroup of $\pi_1(X)$. Then N is normal and we have $\pi_1(X)/N \cong \mathbb{Z}^k \oplus T$ for some finite abelian torsion group T and a nonnegative integer k with $k \leq b_1(X)$. If we set

$$S(\pi_1(X); N) \quad = \quad \{\xi \in S(\pi_1(X)) \,|\, N \leq \ker \xi\}$$

we get that $S(\pi_1(X); N)$ is a subsphere of $S(\pi_1(X))$ of dimension $k - 1$. We will mainly be interested in the case where $T = 0$. This is the case if and only if $N = \ker \xi$ for some $\xi : \pi_1(X) \to \mathbb{R}$.

Example 3.1. Let $E = S^1 \vee S^2$, the one point union of a 1- and a 2-sphere. Furthermore let $X = E \times S^1$. Then $\pi_1(X) \cong \mathbb{Z}^2$ and the universal cover $\tilde{X} = \tilde{E} \times \mathbb{R}$ with $\tilde{E} = \mathbb{R} \cup_{\mathbb{Z}} \bigcup_{n=-\infty}^{\infty} S^2$, the real line with a 2-sphere attached at every integer. $\tilde{E}$ retracts to $\mathbb{R}$ and together with the identity on the other factor this defines a $\mathbb{Z}^2$-equivariant map $h : \tilde{X} \to \mathbb{R}^2$. Furthermore we can define an equivariant homotopy $H : \tilde{X} \times \mathbb{R} \to \tilde{X}$ given by $H(e, t, s) = (e, t + s)$ with $e \in \tilde{E}$, $t, s \in \mathbb{R}$.

If $\xi : \pi_1(X) \to \mathbb{R}$ is a nonzero homomorphism we can find a unique $x_\xi \in S^1$ such that $h_\xi : \tilde{X} \to \mathbb{R}$ given by $h_\xi(x) = \langle x_\xi, h(x) \rangle$ is a control function for ξ. Now H can be used to show that X is ξ-contractible for every ξ such that $x_\xi \neq (\pm 1, 0)$. This means that we get ξ-contractibility as long as ξ does not vanish on $\{1\} \times \pi_1(S^1) \leq \pi_1(E) \times \pi_1(S^1) \cong \pi_1(X)$. Notice that $\tilde{X}$ is not finitely dominated as $H_2(\tilde{X})$ is not finitely generated.

Theorem 3.2. *Let X be a finite connected CW-complex and $N \leq \pi_1(X)$ a normal subgroup such that $\pi_1(X)/N \cong \mathbb{Z}^k$ for some $k \geq 0$. Then X_N is finitely dominated if and only if X is ξ-contractible for all nonzero $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$.*

Alternatively we can say that X_N is finitely dominated if and only if $S(\pi_1(X); N) \subset \Sigma(X)$.

Remark 3.3. For $k = 1$ this theorem is proven in Ranicki [13] as a corollary of [13, Thm.2] which is a chain complex version of Theorem 3.2. We will also give a chain complex version of Theorem 3.2 in Section 4. The geometric version indeed follows from the chain complex version by the work of Wall [19]. We prefer to give a direct proof now as it is fairly elementary.

Remark 3.4. Provided that X is aspherical we get Theorem 3.2 from the work of Bieri and Renz [1, Thm.5.1] by using again Wall [19]. In fact finite domination can be replaced by homotopy finite in the aspherical case.

Proof of Theorem 3.2. The case $k = 0$ is trivial so let us assume that $k \geq 1$.

We start by assuming that X_N is finitely dominated and prove that X is ξ-contractible for all nonzero $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$ by induction.

Assume that $k = 1$. Then $\Delta(X_N : X) \cong \mathbb{Z}$ and let $t : X_N \to X_N$ be a generator of the covering transformation group. Up to multiplication by a positive real number there exist only two nonzero homomorphisms $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$. One with $\xi(t) = 1$ and one with $\xi(t) = -1$. Here we think of ξ as factoring through $\Delta(X_N : X) \cong \pi_1(X)/N$. Let ξ be the homomorphism with $\xi(t) = 1$ and let $a : K \to X_N$, $b : X_N \to K$ satisfy $a \circ b \simeq \mathrm{id}_{X_N}$ with K a finite CW-complex. We can assume that a and b are cellular. By Lemma 2.10 and Proposition 2.8 we get $X \simeq T(bta : K \to K)$ and the homotopy equivalence $\varphi : T_{bta} \to X$ is given by $\varphi([x, s]) = \rho H(tax, s)$, where $\rho : X_N \to X$ is the covering projection and H is a homotopy $H : a \circ b \simeq \mathrm{id}_{X_N}$. Also φ lifts to a $\mathbb{Z}$-equivariant homotopy equivalence $\overline{\varphi} : \overline{T}_{bta} \to X_N$ given by $\varphi([x, t, n]) = t^n H(tax, s)$. From this it is easy to see that $\varphi^* \xi = g_\#$ for the natural projection $g : T_{bta} \to S^1$. As T_{bta} is $g_\#$-contractible by Lemma 2.9 we get that X is ξ-contractible by Lemma 2.4.

To see that X is also $(-\xi)$-contractible replace t by t^{-1}.

Now assume that $k \geq 2$. Let $t_1, \ldots, t_k$ be covering transformations generating $\Delta(X_N : X) \cong \mathbb{Z}^k$ and let $G_1 = \langle t_1 \rangle$ and $G_2 = \langle t_2, \ldots, t_k \rangle$. Furthermore define $X_1 = X_N / G_1$ and $X_2 = X_N / G_2$. So $X_N \to X_1$ is a $\mathbb{Z}$-covering and $X_N \to X_2$ is a $\mathbb{Z}^{k-1}$-covering. Notice that by Lemma 2.11 both X_1 and X_2 are homotopy finite.

Now let $\xi : G \to \mathbb{R}$ be a nonzero homomorphism with $N \leq \ker \xi$. It induces a homomorphism also denoted $\xi : \mathbb{Z}^k \to \mathbb{R}$. We can assume that ξ embeds $\mathbb{Z}^k$ into $\mathbb{R}$ for otherwise we get that X is ξ-contractible by Corollary 2.11 and induction. Let $\xi_i : G_i \to \mathbb{R}$ be defined by $\xi_i = \xi|_{G_i}$ for $i = 1, 2$. Both are injective and they extend to $\mathbb{Z}^k$ and hence to G such that $\xi = \xi_1 + \xi_2$. Without loss of generality we assume that $\xi_1(t_1) = 1$.

Notice that the covering transformation $t_1 : X_N \to X_N$ induces a map $t_1 : X_2 \to X_2$ which generates the covering transformation group $\Delta(X_2 : X) \cong \mathbb{Z}$. We know that X_2 is homotopy finite, so let F be a finite CW-complex and $c : F \to X_2$, $d : X_2 \to F$ be cellular, mutually inverse homotopy equivalences. There is a G_2-covering $\overline{F} \to F$ so that $\overline{F}$ is G_2-equivariantly homotopy equivalent to X_N. Denote $\bar{c} : \overline{F} \to X_N$ and $\bar{d} : X_N \to \overline{F}$ these equivalences. Let $h_2 : \overline{F} \to \mathbb{R}$ be G_2-equivariant, that is $h_2(g_2 x) = h_2(x) + \xi_2(g_2)$ for $x \in \overline{F}$ and $g_2 \in G_2$.

Look at $\bar{d} t_1 \bar{c} : \overline{F} \to \overline{F}$. Then there is a $B > 0$ such that

$$|h_2(\bar{d} t_1 \bar{c}(x)) - h_2(x)| \quad \leq \quad B$$

for all $x \in \overline{F}$, since G_2 acts cocompactly on $\overline{F}$.

Now X_N is finitely dominated, so $\overline{F}$ is finitely dominated as well. By induction there exists a G_2-equivariant map $\bar{q} : \overline{F} \to \overline{F}$ equivariantly homotopic to the identity with

$$h_2 \bar{q}(x) - h_2(x) \quad \leq \quad -B$$

for all $x \in \overline{F}$. Therefore

$$h_2 \bar{q} \bar{d} t_1 \bar{c}(x) - h_2(x) \quad \leq \quad 0 \tag{1}$$

for all $x \in \overline{F}$.

Now $h_2 \bar{q} \bar{d} t_1 \bar{c}, h_2 : \overline{F} \to \mathbb{R}$ are equivariantly homotopic by the straight line homotopy

$H : \overline{F} \times [0,1] \to \mathbb{R}$ with

$$H(x,t) \quad = \quad th_2(x) + (1-t)h_2\bar{q}\bar{d}t_1\bar{c}(x).$$

By Lemma 2.15 we have that $T(\bar{q}\bar{d}t_1\bar{c} : \overline{F} \to \overline{F})$ is a G_2-covering space of $T(qdt_1c : F \to F)$ and we can define a G_2-equivariant map $\bar{h}_2 : T(\bar{q}\bar{d}t_1\bar{c} : \overline{F} \to \overline{F}) \to \mathbb{R}$ by $\bar{h}_2([x,t]) = H(x,t)$, compare Lemma 2.12. It is easy to see that (1) implies

$$\bar{h}_2([\bar{q}\bar{d}t_1\bar{c}(x),t]) - \bar{h}_2([x,t]) \quad \leq \quad 0 \tag{2}$$

Now $T_{\bar{q}\bar{d}t_1\bar{c}} \simeq T_{t_1\bar{c}\bar{q}\bar{d}} \simeq T_{t_1} \simeq X_1$ and all homotopy equivalences are G_2-equivariant equivalences by Lemma 2.14 and Proposition 2.13. It follows that $\overline{T}(\bar{q}\bar{d}t_1\bar{c} : \overline{F} \to \overline{F})$ is $G_1 \times G_2$-equivariant homotopy equivalent to X_N.

Define $\bar{h} : \overline{T}(\bar{q}\bar{d}t_1\bar{c} : \overline{F} \to \overline{F}) \to \mathbb{R}$ by $\bar{h}([x,t,n]) = \bar{h}_2([x,t]) + n + t$ to get a $G_1 \times G_2$ equivariant map, that is we have

$$\begin{aligned}
\bar{h}((t_1^m, g_2)[x,t,n]) &= \bar{h}([g_2x, t, n+m]) \\
&= \bar{h}_2([g_2x,t]) + n + m + t \\
&= \xi_2(g_2) + \bar{h}_2([x,t]) + \xi_1(t_1^m) + n + t \\
&= \bar{h}([x,t,n]) + \xi((t_1^m, g_2)).
\end{aligned}$$

Now define as in Lemma 2.9 $H : \overline{T}_{\bar{q}\bar{d}t_1\bar{c}} \times [0,1] \to \overline{T}_{\bar{q}\bar{d}t_1\bar{c}}$ by

$$H([x,t,n]) \quad = \quad \begin{cases} [x, t-s, n] & t-s \geq 0 \\ [\bar{q}\bar{d}t_1\bar{c}x, 1+t-s, n-1] & t-s \leq 0 \end{cases}$$

Then

$$\begin{aligned}
\bar{h}H_1([x,t,n]) - \bar{h}([x,t,n]) &= \bar{h}([\bar{q}\bar{d}t_1\bar{c}x, t, n-1]) - \bar{h}([x,t,n]) \\
&= \bar{h}_2([\bar{q}\bar{d}t_1\bar{c}x,t]) - \bar{h}_2([x,t]) - 1 \\
&\leq -1
\end{aligned}$$

by (2). By Lemma 2.4 we get that X is ξ-contractible and one direction of the theorem is shown.

It remains to show that X being ξ-contractible for every nonzero $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$ implies that X_N is finitely dominated. We will give a sufficient criterion for a CW-complex Y to be finitely dominated and then show that this criterion is satisfied for X_N.

Let Y be a CW-complex and $H : Y \to \mathbb{R}^k$ a proper map, that is $H^{-1}(C)$ is compact for every compact set $C \subset \mathbb{R}^k$. We denote the Euclidean norm on $\mathbb{R}^k$ by $|\cdot|$.

Lemma 3.5. *Assume there are $R > 0$, $\varepsilon > 0$, $B > 0$, $C > 0$ and a homotopy $K : Y \times [0,1] \to Y$ with $K_0 = \mathrm{id}_Y$ such that*

$$\varepsilon \leq |H(x)| - |HK_1(x)| \leq C \tag{3}$$

for all $x \in Y$ with $|H(x)| \geq R$ and

$$-B \leq |H(x)| - |HK(x,t)| \leq C \tag{4}$$

for all $x \in Y$ with $|H(x)| \geq R$ and all $t \in [0,1]$. Then Y is finitely dominated.

Proof. We can assume that $\varepsilon \geq 2B$. For otherwise we define

$$\overline{K}(x,t) \quad = \quad \begin{cases} K(x,2t) & 0 \leq t \leq \frac{1}{2} \\ K(K(x,2t-1),1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then

$$\begin{aligned} |H(x)| - |H\overline{K}_1(x)| &= |H(x)| - |HK_1(x)| + |HK_1(x)| - |HK_1(K_1(x))| \\ &\in [2\varepsilon, 2C] \end{aligned}$$

for all $x \in Y$ with $|H(x)| \geq R + C$ and

$$|H(x)| - |H\overline{K}(x,t)| = |H(x)| - |HK(x,2t)| \in [-B, C]$$

or

$$\begin{aligned} &= |H(x)| - |HK(x,2t-1)| + |HK(x,2t-1)| - |HK_1(K(x,2t-1))| \\ &\in [-B+\varepsilon, 2C] \end{aligned}$$

for all $x \in Y$ with $|H(x)| \geq R + C$ and $t \in [0,1]$. Therefore we can increase the ε by increasing C and R.

So now assume that

$$|H(x)| - |HK_1(x)| \geq 2B \tag{5}$$

for all $x \in Y$ with $|H(x)| \geq R$.

Since $\{x \in Y \mid |H(x)| \leq R\}$ is compact, there is an $L \geq R+B$ such that $|HK_1(x)| \leq L$ for all $x \in Y$ with $|H(x)| \leq R$.

Let $\lambda : Y \to [0,1]$ be a map with $\lambda(x) = 0$ for $x \in Y$ with $|H(x)| \leq L$ and $\lambda(x) = 1$ for $x \in Y$ with $|H(x)| \geq L + B$. We get the following properties

(i) If $|H(x)| \leq R + (2n+1)B$ then $\lambda(K_1^n(x)) = 0$ for all $n \geq 0$.
(ii) If $\lambda(K_1(x)) > 0$, then $\lambda(x) = 1$.
(iii) If $\lambda(x) < 1$, then $\lambda(K_1(x)) = 0$.

Therefore the sequence $(\lambda(K_1^l(x)))_{l=0}^\infty$ is monotonely decreasing with at most one term in $(0,1)$ and only finitely many terms bigger than 0 for every $x \in Y$.

We prove (i) by induction: For $n = 0$ we get $\lambda(x) = 0$ since $L \geq R + B$. So assume $n \geq 1$. If $|H(x)| \in [R + (2n-1)B, R + (2n+1)B]$, then $|HK_1(x)| \leq |H(x)| - 2B \leq R + (2n-1)B$ by (5). Thus $\lambda(K_1^{n-1}(K_1(x))) = 0$ by induction. If $|H(x)| \leq R + (2n-1)B$, then $\lambda(K_1^{n-1}(x)) = 0$ and $\lambda(K_1^n(x)) = 0$ follows from (iii).

To see (iii) note that $\lambda(x) < 1$ implies $|H(x)| < L + B$. If $|H(x)| \leq R$, then $|HK_1(x)| \leq L$ by the choice of L. If $|H(x)| \geq R$, then

$$|HK_1(x)| \leq |H(x)| - 2B < L - B$$

by (5). In both cases we get $\lambda(K_1(x)) = 0$.

To prove (ii) note that $\lambda(K_1(x)) > 0$ implies $|HK_1(x)| > L$, so $|H(x)| \geq R$. Now

$$|H(x)| \geq 2B + |HK_1(x)| > 2B + L$$

and therefore $\lambda(x) = 1$.

Let us define homotopies $\mathcal{K}^n : Y \times [0,1] \to Y$ by

$$\begin{aligned} \mathcal{K}^0(x,t) &= K(x, t \cdot \lambda(x)) \\ \mathcal{K}^n(x,t) &= K(\mathcal{K}^{n-1}(x,1), t \cdot \lambda(K_1^n(x))) \end{aligned}$$

Notice that $\mathcal{K}^n(x,0) = \mathcal{K}^{n-1}(x,1)$ and $\mathcal{K}^0(x,0) = x$. Also for $|H(x)| \leq R + (2n+1)B$ we get for $m \geq n$

$$\mathcal{K}^m(x,t) = \mathcal{K}^{n-1}(x,1)$$

so the homotopies become eventually constant for fixed $x \in Y$. Therefore they combine to a homotopy $\mathcal{K} : Y \times [0,1] \to Y$ with $\mathcal{K}(x,0) = x$ and $\mathcal{K}(x,1) = \mathcal{K}^{n-1}(x,1)$ where n is a positive integer with $|H(x)| \leq R + (2n+1)B$. Also

$$\begin{aligned} \mathcal{K}^{n-1}(x,1) &= K(K(\dots K(x,\lambda(x)),\lambda(K_1(x))\dots),\lambda(K_1^{n-1}(x))) \\ &= K(K_1^{k_x}(x),\lambda(K_1^{k_x}(x))) \end{aligned}$$

where k_x is the minimal nonnegative integer with $\lambda(K_1^{k_x+1}(x)) = 0$. This means $|HK_1^{k_x+1}(x)| \leq L$. If $|HK_1^{k_x}(x)| \geq R$ then

$$|HK_1^{k_x}(x)| \quad \leq \quad C + L$$

and

$$|HK(K_1^{k_x}(x),\lambda(K_1^{k_x}(x)))| \leq C + L + B$$

by (4).

If $|HK_1^{k_x}(x)| \leq R$, then $\lambda(K_1^{k_x}(x)) = 0$ (which by the definition of k_x implies $k_x = 0$) and so

$$|HK(K_1^{k_x}(x),\lambda(K_1^{k_x}(x)))| \quad \leq \quad R.$$

Therefore the image of $\mathcal{K}_1$ is contained in a compact subset of Y and so Y is dominated by a finite subcomplex. □

Let us return to the proof of Theorem 3.2. $\mathbb{Z}^k$ acts on X_N by covering transformations. It also acts on $\mathbb{R}^k$ by translation. Since $\mathbb{R}^k$ is contractible we define an equivariant map $H : X_N \to \mathbb{R}^k$. Since X is a finite CW-complex we get that $\mathbb{Z}^k$ acts cocompactly on X_N and therefore H is a proper map.

For every $\xi \in S(\pi_1(X); N)$ there is a unique $x_\xi \in S^{k-1} \subset \mathbb{R}^k$ such that

$$h_\xi(x) \quad = \quad \langle x_\xi, H(x) \rangle$$

is a control function for ξ. We assume that for every $\xi \in S(\pi_1(X); N)$ we have that X is ξ-contractible. Hence given an $\varepsilon > 0$ there exist equivariant homotopies $H_\xi : X_N \times [0,1] \to X_N$ with $H_{\xi 0} = \mathrm{id}_{X_N}$ and

$$h_\xi H_\xi(x,1) - H_\xi(x) \quad \leq \quad -\varepsilon$$

for all $x \in X_N$.

We will now identify $S(\pi_1(X); N)$ with $S^{k-1} \subset \mathbb{R}^k$ via $\xi \leftrightarrow x_\xi$. As in the proof of Proposition 2.6 we get that for every $\xi \in S^{k-1}$ there exists a neighborhood U of ξ in S^{k-1} such that for every $\xi' \in U$ we have

$$h_{\xi'} H_\xi(x,1) - h_{\xi'}(x) \quad \leq \quad -\frac{\varepsilon}{2}$$

for all $x \in X_N$. In other words we can use the same H_ξ for all ξ' in a small neighborhood of ξ. By compactness there exist finitely many $\xi_1, \dots, \xi_m$ and $R_1, \dots, R_m > 0$ so that for $i = 1, \dots, m$

$$U_i \quad = \quad \{\xi \in S^{k-1} \,|\, |\xi - \xi_i| < R_i\}$$

cover S^{k-1} and for all $\xi \in U_i$ we get

$$h_\xi H_{\xi_i}(x,1) - h_\xi(x) \quad \leq \quad -\frac{\varepsilon}{2} \tag{6}$$

for all $x \in X_N$.

Again by compactness there is an $A \geq 0$ such that

$$h_\xi H_{\xi_i}(x,t) - h_\xi(x) \quad \leq \quad A \tag{7}$$

for all $x \in X_N$, $t \in [0,1]$ and $\xi \in \overline{U}_i$, the closure of U_i.

By iterating the homotopies H_{ξ_i} with themselves we can increase the ε in (6) without increasing the A in (7). Therefore we can assume that

$$\begin{aligned} h_\xi H_{\xi_i}(x,1) - h_\xi(x) &\leq -6(m+1)A \\ h_\xi H_{\xi_i}(x,t) - h_\xi(x) &\leq A \end{aligned}$$

for all $x \in X_N$, $t \in [0,1]$ and $\xi \in U_i$, $i = 1, \ldots, m$.

Let $B = 6(m+1)A$ and

$$CU_i \quad = \quad \{x \in \mathbb{R}^k - \{0\} \,|\, \frac{x}{|x|} \in U_i\}.$$

Then

$$\bigcup_{i=1}^m CU_i \quad = \quad \mathbb{R}^k - \{0\}.$$

By compactness there is a $C > 0$ such that

$$|H(x) - HH_{\xi_i}(x,t)| \quad \leq \quad C$$

for all $x \in X_N$, $t \in [0,1]$ and $i = 1, \ldots, m$. Notice that necessarily we have $C \geq B$.

Now for $x \in H^{-1}(CU_i)$ let $\xi = \frac{H(x)}{|H(x)|} \in U_i$. Then

$$\begin{aligned} |HH_{\xi_i}(x,t)|^2 - |H(x)|^2 &= |HH_{\xi_i}(x,t) - H(x) + H(x)|^2 - |H(x)|^2 \\ &= |HH_{\xi_i}(x,t) - H(x)|^2 + 2\langle HH_{\xi_i}(x,t) - H(x), H(x)\rangle \\ &\leq C^2 + 2|H(x)|(h_\xi(H_{\xi_i}(x,t)) - h_\xi(x)) \end{aligned}$$

So

$$|HH_{\xi_i}(x,t)| - |H(x)| \quad \leq \quad \frac{C^2 + 2|H(x)|(h_\xi H_{\xi_i}(x,t) - h_\xi(x))}{|HH_{\xi_i}(x,t)| + |H(x)|}$$

So for arbitrary $t \in [0,1]$ and $|H(x)| \geq \frac{C^2}{A}$ we get

$$|HH_{\xi_i}(x,t)| - |H(x)| \quad \leq \quad 3A.$$

For $t = 1$ we get

$$|HH_{\xi_i}(x,1)| - |H(x)| \quad \leq \quad A - \frac{2|H(x)|(h_\xi H_{\xi_i}(x,1) - h_\xi(x))}{|HH_{\xi_i}(x,1)| + |H(x)|}$$

Now $|HH_{\xi_i}(x,1)| \leq C + |H(x)|$ and since $\frac{C^2}{A} \geq C$ we get

$$|HH_{\xi_i}(x,1)| + |H(x)| \quad \leq 3|H(x)|$$

and therefore

$$|HH_{\xi_i}(x,1)| - |H(x)| \quad \leq \quad A - \frac{2}{3}B \quad = \quad -(4m+3)A.$$

We know that the U_i cover S^{k-1}. Also there is a $\delta > 0$ such that if we define $V_i = \{\xi \in U_i \,|\, |\xi - \xi_i| < R_i - \delta\}$, the V_i still cover S^{k-1}. Also let $W_i = \{\xi \in S^{k-1} \,|\, |\xi - \xi_i| = R_i\}$, the boundary of U_i.

Now let $\lambda_i : \mathbb{R}^k \to [0,1]$ be a map such that $\lambda_i(x) = 0$ for $x \in \mathbb{R}^k - CU_i$ and for $|x| \leq \frac{C^2}{A}$. Also we want $\lambda_i(x) = 1$ for $x \in CU_i$ with $|x| \geq \frac{C^2}{A} + 1$ and $|x - y| \geq \delta$

for all $y \in CW_i = \{x \in \mathbb{R}^k - \{0\} \mid \frac{x}{|x|} \in W_i\}$.

Now define $K_i : X_N \times [0,1] \to X_N$ by

$$K_i(x,t) = H_{\xi_i}(x, t \cdot \lambda_i(H(x)))$$

Then $K_i(x,0) = x$ for all $x \in X_N$ and we have

$$|HK_i(x,1)| - |H(x)| \quad \leq \quad -(4m+3)A$$

for $x \in H^{-1}(CU_i)$ with $|H(x)| \geq \frac{C^2}{A} + 1$ and $|H(x) - y| \geq \delta$ for all $y \in CW_i$. Also

$$|HK_i(x,t)| - |H(x)| \quad \leq \quad 3A$$

for all $x \in X_N$.

Since the V_i cover S^{k-1} we get that every $x \in X_N$ with $|H(x)| \geq \frac{C^2}{A} + 1$ lies in at least one $H^{-1}(CU_i)$ and satisfies $|H(x) - y| \geq \delta$ for all $y \in CW_i$.

Define inductively

$$\begin{aligned} \mathcal{K}_1(x,t) &= K_1(x,t) \\ \mathcal{K}_{i+1}(x,t) &= K_{i+1}(\mathcal{K}_i(x,1),t). \end{aligned}$$

These homotopies combine to a homotopy $\mathcal{K} : X_N \times [0,m] \to X_N$ where $\mathcal{K}|X_N \times [i,i+1] = \mathcal{K}_{i+1}$. We claim that $\mathcal{K}$ has the properties required for Lemma 3.5. Let $x \in X_N$ satisfy $|H(x)| \geq \frac{C^2}{A} + 1 + \frac{3mC}{\delta}$. Then there is an i such that $H(x) \in CV_i$ and we have $|H(x) - y| \geq 3mC$ for all $y \in CW_i$. Since $|HK_j(x,t) - H(x)| \leq C$ this implies that $H\mathcal{K}(x,i-1) \in CU_i$ and $|H\mathcal{K}(x,i-1)| \geq \frac{C^2}{A} + 1$. In general we have $|H\mathcal{K}_j(x,1)| - |H\mathcal{K}_{j-1}(x,1)| \leq 3A$. Therefore

$$\begin{aligned} |H\mathcal{K}(x,i)| - |H(x)| &= |H\mathcal{K}_i(x,1)| - |H\mathcal{K}_{i-1}(x,1)| + |H\mathcal{K}_{i-1}(x,1)| - |H(x)| \\ &\leq |HK_i(\mathcal{K}_{i-1}(x,1),1)| - |H\mathcal{K}_{i-1}(x,1)| + 3(i-1)A \\ &\leq (-4m+3)A + 3(i-1)A \end{aligned}$$

Also

$$|H\mathcal{K}_m(x,1)| - |H\mathcal{K}_i(x,1)| \quad \leq 3(m-i)A$$

so

$$|H\mathcal{K}(x,m)| - |H(x)| \quad \leq \quad -A$$

and we can choose $\varepsilon = A$. The remaining bounds for Lemma 3.5 are achieved since the K_i are built out of the equivariant H_{ξ_i} and $\mathcal{K}$ is built from the K_i in a finite number of steps. So we get the remaining constants from compactness arguments. Therefore X_N is finitely dominated by Lemma 3.5 and this finishes the proof of Theorem 3.2. □

4. Relations with Novikov homology

Let R be a ring with unit. We denote by R^G the abelian group of all functions $\lambda : G \to R$. For $\lambda \in R^G$ denote $\operatorname{supp} \lambda = \{g \in G \mid \lambda(g) \neq 0\}$.

Definition 4.1. Let $\xi : G \to \mathbb{R}$ be a homomorphism. The *Novikov ring* $\widehat{RG}_\xi$ is defined as

$$\widehat{RG}_\xi \quad = \quad \{\lambda \in R^G \mid \forall\, r \in \mathbb{R} \quad \operatorname{supp} \lambda \cap \xi^{-1}([r,\infty)) \text{ is finite}\}$$

with $\lambda \cdot \mu(g) = \sum \lambda(g_1)\mu(g_2)$ for $\lambda, \mu \in \widehat{RG}_\xi$. The sum is taken over all $g_1, g_2 \in G$ with $g_1 g_2 = g$.

For $\lambda \in \widehat{RG}_\xi$ let

$$\|\lambda\|_\xi \quad = \quad \inf\{t \in (0,\infty) \,|\, \operatorname{supp}\lambda \subset \xi^{-1}((-\infty, \log t])\}$$

be the *norm* of λ with respect to ξ. Note that $\widehat{RG}_\xi$ is a completion of the group ring RG with respect to the metric induced by this norm. We can extend the definition of the norm to $n \times m$ matrices over $\widehat{RG}_\xi$ by setting

$$\|A\|_\xi \quad = \quad \max\{\|A_{ij}\|_\xi \,|\, i \in \{1,\dots,n\}, j \in \{1,\dots,m\}\}\,.$$

It is easy to see that

$$\|A \cdot B\|_\xi \quad \le \quad \|A\|_\xi \cdot \|B\|_\xi \tag{8}$$

for an $n \times m$ matrix A and an $m \times k$ matrix B.

Now let X be a finite CW-complex and $\tilde{X}$ its universal cover. We set $G = \pi_1(X)$. The cellular complex $C_*(\tilde{X})$ is a finitely generated free $\mathbb{Z}G$-chain complex and we also denote it by $C_*(X;\mathbb{Z}G)$. Furthermore if $\varepsilon : \mathbb{Z}G \to R$ is a ring homomorphism, we set

$$C_*(X;R) \quad = \quad R \otimes_{\mathbb{Z}G} C_*(\tilde{X})$$

The corresponding homology will be denoted by $H_*(X;R)$.

The vanishing of the *Novikov homology* $H_*(X;\widehat{\mathbb{Z}G}_\xi)$ is closely related to ξ-contractibility of X for we have

Proposition 4.2. *Let X be ξ-contractible, then $H_*(X;\widehat{\mathbb{Z}G}_\xi) = 0$.*

Proof. See Latour [8, Prop.1.10]. □

The converse holds provided that the homomorphism ξ satisfies a stability condition which stems from the work of Bieri and Renz [1], see Latour [8, Cor.5.23]. It was shown by Damian [2] that the vanishing of the Novikov homology alone does not imply ξ-contractibility in general.

If $C_*(X;\widehat{\mathbb{Z}G}_\xi)$ is acyclic we are also interested in its torsion. For this define

$$\mathrm{Wh}(G;\xi) \quad = \quad K_1(\widehat{\mathbb{Z}G}_\xi)/\langle \tau(\pm g), \tau(1-a) \,|\, g \in G, \|a\|_\xi < 1\rangle.$$

An acyclic Novikov complex $C_*(X;\widehat{\mathbb{Z}G}_\xi)$ then defines a well defined torsion

$$\tau(X,\xi) \quad = \quad \tau(C_*(X;\widehat{\mathbb{Z}G}_\xi)) \in \mathrm{Wh}(G;\xi)$$

Analogously to $\Sigma(X)$ we define

$$\begin{aligned}\Sigma(X;\mathbb{Z}) \quad &= \quad \{\xi \in S(G) \,|\, H_*(X;\widehat{\mathbb{Z}G}_\xi) = 0\} \\ \Sigma_s(X;\mathbb{Z}) \quad &= \quad \{\xi \in \Sigma(X;\mathbb{Z}) \,|\, \tau(X,\xi) = 0\}\end{aligned}$$

The following Proposition is already stated in Latour [8, Prop.1.17], but as is pointed out in Damian [2, §2.3] the proof there is not correct. Damian gives an alternative proof of the statement about $\Sigma(X;\mathbb{Z})$ in the case that X is homotopy equivalent to a closed manifold.

Proposition 4.3. *Let X be a finite CW-complex. Then $\Sigma(X;\mathbb{Z})$ and $\Sigma_s(X;\mathbb{Z})$ are open subsets of $S(\pi_1(X))$.*

Proof. Let $\xi \in \Sigma(X;\mathbb{Z})$. Since $C_*(X;\widehat{\mathbb{Z}G}_\xi)$ is finitely generated free, there exists a chain contraction $\delta : C_*(X;\widehat{\mathbb{Z}G}_\xi) \to C_{*+1}(X;\widehat{\mathbb{Z}G}_\xi)$. Choose a basis of $C_*(X;\mathbb{Z}G)$ by choosing liftings and orientations for every cell of X. Then δ is represented by a matrix Δ with entries in $\widehat{\mathbb{Z}G}_\xi$. The boundary operator can also be represented by a matrix which we denote by ∂. Therefore

$$\partial\Delta + \Delta\partial \quad = \quad I.$$

Now we define a matrix $\overline{\Delta}$ with entries in $\mathbb{Z}G$ according to the following rule:

$$\overline{\Delta}_{ij}(g) \quad = \quad \begin{cases} \Delta_{ij}(g) & \text{if } \xi(g) \geq \log(\|\partial\|_\xi^{-1}) \\ 0 & \text{otherwise} \end{cases}$$

Then $\|\Delta - \overline{\Delta}\|_\xi < \|\partial\|_\xi^{-1}$ and we have

$$\begin{aligned} \partial\overline{\delta} + \overline{\Delta}\partial &= \partial(\overline{\Delta} - \Delta) + \partial\Delta + (\overline{\Delta} - \Delta)\partial + \Delta\partial \\ &= I + \partial(\overline{\Delta} - \Delta) + (\overline{\Delta} - \Delta)\partial \\ &= I + A \end{aligned}$$

where A is a matrix with $\|A\|_\xi < 1$. Also A is a matrix over $\mathbb{Z}G$ since I, $\partial\overline{\Delta}$ and $\overline{\Delta}\partial$ are matrices over $\mathbb{Z}G$.

Since the topology on $S(\pi_1(X))$ can be thought of as the compact-open topology we get a neighborhood U of ξ in $S(\pi_1(X))$ such that $\|A\|_{\xi'} < 1$ for all $\xi' \in U$. In particular $(I + A)^{-1}$ is a well defined matrix over $\widehat{\mathbb{Z}G}_{\xi'}$ for all $\xi' \in U$. But the matrix $\overline{\Delta}(I+A)^{-1}$ defines a chain contraction $\delta_{\xi'} : C_*(X;\widehat{\mathbb{Z}G}_{\xi'}) \to C_{*+1}(X;\widehat{\mathbb{Z}G}_{\xi'})$ for all $\xi' \in U$ and therefore $\Sigma(X;\mathbb{Z})$ is open.

To see that $\Sigma_s(X;\mathbb{Z})$ is open let us assume that $\tau(X,\xi) = 0 \in \mathrm{Wh}(G;\xi)$. By choosing an appropriate basis by liftings of cells we can assume that

$$\tau(X,\xi) \quad = \quad 0 \in K_1(\widehat{\mathbb{Z}G}_\xi)/\langle \tau(1-a) \,|\, \|a\|_\xi < 1\rangle$$

By the definition of torsion $\tau(X,\xi)$ is represented by the matrix $(\partial + \Delta)$ which represents an isomorphism $\partial + \delta : C_{\mathrm{odd}}(X;\widehat{\mathbb{Z}G}_\xi) \to C_{\mathrm{even}}(X;\widehat{\mathbb{Z}G}_\xi)$ with $C_{\mathrm{odd}} = \bigoplus_{n\in\mathbb{Z}} C_{2n+1}$ and $C_{\mathrm{even}} = \bigoplus_{n\in\mathbb{Z}} C_{2n}$, ∂ the boundary operator and δ a chain contraction.

Now as seen in the first part the matrix $\partial + \Delta$ can be chosen to be $\partial + \overline{\Delta}(I+A)^{-1}$ with ∂, $\overline{\Delta}$ and A matrices over $\mathbb{Z}G$. Furthermore there is a neighborhood U of ξ in $S(\pi_1(X))$ such that $\|A\|_{\xi'} < 1$ for all $\xi' \in U$. By Lemma 4.4 below we get

$$\begin{pmatrix} \partial + \overline{\Delta}(I+A)^{-1} & \\ & I \end{pmatrix} \quad = \quad E_1 \cdots E_k \cdot (I - E) \tag{9}$$

with the E_j elementary matrices over $\mathbb{Z}G$ and a matrix E over $\widehat{\mathbb{Z}G}_\xi$ with $\|E\|_\xi < 1$. We recall that an elementary matrix over a ring R with unit is an $n \times n$ matrix E^x_{ij} for $i \neq j$ and $x \in R$ which has 1 in every diagonal spot, x in the (i,j) spot and zero everywhere else.

Now (9) gives

$$(E_1 \cdots E_k)^{-1} \begin{pmatrix} \partial(I+A) + \overline{\Delta} & \\ & I \end{pmatrix} \quad = \quad (I - E) \begin{pmatrix} I + A & \\ & I \end{pmatrix}$$

The right side is a matrix of the form $I - B$ with $\|B\|_\xi < 1$ and the entries of B are in $\mathbb{Z}G$ because the left side is a matrix over $\mathbb{Z}G$. Thus there is a small neighborhood

V of ξ so that $\|B\|_{\xi'} < 1$ for all $\xi' \in V$. But then $\|E\|_{\xi'} < 1$ for all $\xi' \in V \cap U$, a neighborhood of ξ. Now $\tau(X, \xi') = 0 \in \mathrm{Wh}(G; \xi')$ for all $\xi' \in U \cap V$ because of (9). This finishes the proof modulo Lemma 4.4. □

Lemma 4.4. *Let A be an invertible $n \times n$ matrix over $\widehat{RG}_\xi$ with $\tau(A) = 0 \in K_1(\widehat{RG}_\xi)/\langle \tau(1-a) \,|\, \|a\|_\xi < 1 \rangle$. Then there exist elementary matrices $E_1, \ldots, E_k$ over RG and a matrix E over $\widehat{RG}_\xi$ with $\|E\|_\xi < 1$ such that for a stabilization of A we get*

$$\begin{pmatrix} A & \\ & I \end{pmatrix} = E_1 \cdots E_k \cdot (I - E)$$

Proof. Since $i_* \tau(A) = 0$ we get $\begin{pmatrix} A & \\ & I \end{pmatrix} = F_1 \cdots F_l$ with the F_i being either elementary matrices over $\widehat{RG}_\xi$ or matrices of the form $I - D$ with $\|D\|_\xi < 1$. Since the elementary matrices generate the commutator of $\mathrm{GL}(R)$ for any ring R with unit we can assume that $F_l = I - D$ with $\|D\|_\xi < 1$ and the remaining matrices are elementary.

It remains to show that we can replace the elementary matrices over $\widehat{RG}_\xi$ by elementary matrices over RG. For this we will prove the following:

Given elementary matrices $E'_1, \ldots, E'_k$ over $\widehat{RG}_\xi$ and $\varepsilon \in (0,1)$, there exist elementary matrices $E_1, \ldots, E_k$ over RG and a matrix E over RG with $\|E\|_\xi < \varepsilon$, such that

$$E'_1 \cdots E'_k = E_1 \cdots E_k \cdot (I - E) \tag{10}$$

We prove it by induction on k. The case $k = 0$ is trivial. Now assume the statement is true for $k - 1$. Then $E'_1 \cdots E'_k = E'_1 \cdots E'_{k-1} \cdot E'_k$. By induction hypothesis we can find elementary matrices $E_1, \ldots, E_{k-1}$ over RG and E' with $\|E'\|_\xi < \varepsilon \cdot \|E'_k\|_\xi^{-2}$ such that $E'_1 \cdots E'_{k-1} = E_1 \cdots E_{k-1} \cdot (I - E')$. Now

$$(I - E') \cdot E'_k = E'_k \cdot (I - (E'_k)^{-1} \cdot E' \cdot E'_k).$$

Since we can write $E'_k = E_k - R_k = E_k(I - E_k^{-1} R_k)$ with E_k an elementary matrix over RG and $\|R_k\|_\xi < \varepsilon \cdot \|E'_k\|_\xi^{-1}$ we get the claim. Notice that $\|E'_k\|_\xi^{-1} = \|E_k\|_\xi^{-1} = \|E_k^{-1}\|_\xi^{-1}$.

This shows (10) and the lemma follows. □

As promised in Remark 3.3 we now want to get to a chain complex version of Theorem 3.2. So let R be a ring with unit and (C_*, d) a chain complex over R. We will always assume that $C_i = 0$ for negative integers i. The chain complex C_* is *finitely generated*, if there is an $n \in \mathbb{Z}$ such that $C_i = 0$ for $i \geq n$ and every C_i is a finitely generated R-module. A chain complex C_* is *free*, respectively *projective*, if for every $i \in \mathbb{Z}$ C_i is a free R-module, respectively a projective R-module.

Definition 4.5. A chain complex C_* over R is *finitely dominated*, if there exist a finitely generated free R-chain complex D_*, chain maps $a : D_* \to C_*$, $b : C_* \to D_*$ and a chain homotopy $H : C_* \to C_{*+1}$ with $H : ab \simeq \mathrm{id}_C$.

Proposition 4.6. *An R-chain complex C_* is finitely dominated if and only if it is chain homotopy equivalent to a finitely generated projective R-chain complex D_*.*

Proof. See Ranicki [12, Prop.3.2]. □

Now let G be a group and $N \leq G$ a normal subgroup such that $G/N \cong \mathbb{Z}^k$ for some $k \geq 0$. Inclusion gives a ring homomorphism $i : RN \to RG$. For a (left) RN-module M we get a (left) RG-module by $i_! M = RG \otimes_{RN} M$. Also if M is an RG-module, we denote $i^! M$ to be the RN-module obtained by restriction. Notice that if M is a finitely generated RG-module, $i^! M$ will in general not be finitely generated.

Theorem 4.7. *Let R be a ring with unit, G a group, $N \leq G$ a normal subgroup such that $G/N \cong \mathbb{Z}^k$ for some $k \geq 0$ and C_* a finitely generated free RG-chain complex. Then the free RN-chain complex $i^! C_*$ is finitely dominated if and only if $\widehat{RG}_\xi \otimes_{RG} C_*$ is acyclic for all nonzero $\xi : G \to \mathbb{R}$ with $N \leq \ker \xi$.*

Before we give the proof of Theorem 4.7 let us give a criterion to decide when $\widehat{RG}_\xi \otimes_{RG} C_*$ is an acyclic chain complex. Choose a basis $\mathcal{B}$ of the finitely generated free RG-chain complex C_* which is a disjoint union of finite sets $\mathcal{B}_i$, each being a basis for C_i, $i \in \mathbb{Z}$. Then every $y \in C_i$ can be written as

$$y \quad = \quad \sum_{x \in \mathcal{B}_i} y_x x$$

with $y_x \in RG$ and we define

$$\|y\|_\xi \quad = \quad \max\{\|y_x\|_\xi \in [0, \infty) \,|\, x \in \mathcal{B}_i\}$$

Obviously $\| \cdot \|_\xi$ depends on the basis so we will also write $\| \cdot \|_\xi^{\mathcal{B}}$ if we want to indicate this.

Lemma 4.8. *Let C_* be a finitely generated free RG-chain complex, $\mathcal{B}$ a basis of C_* and $\xi : G \to \mathbb{R}$ a homomorphism. Then $\widehat{RG}_\xi \otimes_{RG} C_*$ is acyclic if and only if there is an RG-chain map $a : C_* \to C_*$ chain homotopic to the identity such that*

$$\|a(x)\|_\xi \quad < \quad 1$$

for every $x \in \mathcal{B}$.

Proof. Assume that $\widehat{RG}_\xi \otimes_{RG} C_*$ is acyclic. As in the proof of Proposition 4.3 we can define a chain homotopy $H : C_* \to C_{*+1}$ with $\partial H + H\partial = 1 - a$ so that $\|a(x)\|_\xi < 1$ for all $x \in \mathcal{B}$ by modifying a chain contraction $\delta : \widehat{RG}_\xi \otimes_{RG} C_* \to \widehat{RG}_\xi \otimes_{RG} C_{*+1}$. Now a is a chain map and chain homotopic to the identity.

If we assume the existence of the chain map $a : C_* \to C_*$ chain homotopic to the identity with $\|a(x)\|_\xi < 1$ for every $x \in \mathcal{B}$, then $1 + a + a^2 + \ldots : \widehat{RG}_\xi \otimes_{RG} C_* \to \widehat{RG}_\xi \otimes_{RG} C_*$ is a well defined chain map and the inverse of the chain map $1 - a$. It follows that $1 - a$ induces an isomorphism of the homology $H_*(\widehat{RG}_\xi \otimes_{RG} C_*)$, but it also induces the zero map on this homology since $1 - a$ is chain homotopic to the zero map. Thus $\widehat{RG}_\xi \otimes_{RG} C_*$ is acyclic. □

The proof of Theorem 4.7 is analogous to the proof of Theorem 3.2, we mainly have to show how to carry over the geometric arguments to arguments dealing with chain complexes.

Proof of Theorem 4.7. We need chain complex analogues for the geometric constructions in Section 2 and 3. Let us start again by assuming that $i^! C_*$ is finitely dominated. We need to show that $\widehat{RG}_\xi \otimes_{RG} C_*$ is acyclic for all nonzero $\xi : G \to \mathbb{R}$ with $N \leq \ker \xi$.

Assume $k = 1$. Then G is the semidirect product $G = N \times_\alpha \mathbb{Z}$ with $\alpha : N \to N$ the automorphism induced by conjugation with $t \in G$ so that its projection in $G/N \cong \mathbb{Z}$ is a generator. Up to multiplication by a positive real number there are only two homomorphism we have to consider, namely $\xi : G \to \mathbb{R}$ such that $\xi(t) = 1$ and $-\xi$.

Define $\zeta : i^! C_* \to i^! C_*$ by $\zeta(x) = tx$ with $t \in G$ as above. Then ζ commutes with the boundary, but it is in general not an RN map since $\zeta(hx) = \alpha(h)\zeta(x)$ for $h \in N$. But we get an RG-chain map

$$\begin{array}{rcl} 1 - t^{-1} \otimes \zeta : i_! i^! C_* & \longrightarrow & i_! i^! C_* \\ 1 \otimes x & \mapsto & 1 \otimes x - t^{-1} \otimes tx \end{array}$$

In analogy with Ranicki [13] we define the *mapping torus* of ζ

$$T(\zeta) \quad = \quad \mathcal{C}(1 - t^{-1} \otimes \zeta : i_! i^! C_* \to i_! i^! C_*)$$

as the mapping cone of $1 - t^{-1} \otimes \zeta$. This mapping torus has the analogues properties of the geometric one: the projection $p : T(\zeta) \to C_*$ given by $p(1 \otimes x, 1 \otimes y) = x$ is a chain homotopy equivalence. Furthermore $T(\zeta)$ is chain homotopy equivalent to $T(b\zeta a)$, where $a : D_* \to i^! C_*$, $b : i^! C_* \to D_*$ are mutually inverse chain homotopy equivalences between $i^! C_*$ and a finitely generated projective RN-chain complex D_* which exists since we assumed $i^! C_*$ to be finitely dominated.

But $\widehat{RG}_\xi \otimes_{RG} T(b\zeta a)$ is acyclic for $\xi(t) = 1$ since $1 - t^{-1} \otimes b\zeta a : \widehat{RG}_\xi \otimes_{RG} i_! D_* \to \widehat{RG}_\xi \otimes_{RG} i_! D_*$ is an automorphism in every degree, compare also the proof of Theorem 2 in Ranicki [13].

So now assume that $k \geq 2$. Let $H \leq G$ be a subgroup containing N such that $G/H \cong \mathbb{Z}^l$ and $H/N \cong \mathbb{Z}^{k-l}$ with $l \geq 1$. Let $j_1 : RN \to RH$ and $i_1 : RH \to RG$ be the inclusions. As in Corollary 2.11 we get that $i_1^! C_*$ is chain homotopy equivalent to a finitely generated free RH-chain complex. Notice that we get that $i_1^! C_* \simeq D_*$ with D_* a finitely generated projective RH-chain complex and D_* is a mapping torus. Therefore $[D_*] = 0 \in \widetilde{K}_0(RH)$ and by Ranicki [12] D_* is chain homotopy equivalent to a finitely generated free RH-chain complex F_*.

So now choose H such that $G/H \cong \mathbb{Z}$ and $H/N \cong \mathbb{Z}^{k-1}$. Let $g \in G$ project to a generator of G/H and write $\xi = \xi_1 + \xi_2$, where ξ_1 vanishes on H, ξ_2 vanishes on N and $\xi_2(g) = 0$. We can assume that $G/\ker \xi_1 \cong \mathbb{Z}$ and $G/\ker \xi_2 \cong \mathbb{Z}^{k-1}$ for otherwise we get that $\widehat{RG}_\xi \otimes_{RG} C_*$ is acyclic by induction.

Now $j_1^! F_* \simeq j_1^! i_1^! C_* = i^! C_*$ is a finitely dominated RN-chain complex. Let $\varepsilon > 0$. By induction and Lemma 4.8 there is a chain map $a : F_* \to F_*$ chain homotopic to the identity with

$$\|a(x)\|_{\xi_2} \quad \leq \quad \varepsilon \cdot \|x\|_{\xi_2}.$$

for every $x \in F_*$. Now $C_* \simeq T(d\zeta ca : i_{1!} F_* \to i_{1!} F_*)$. By choosing the $\varepsilon > 0$ small enough we see that $\widehat{RG}_\xi \otimes_{RG} C_* \simeq 0$ by the same argument as in the case $k = 1$.

Now we assume that $\widehat{RG}_\xi \otimes_{RG} C_*$ is acyclic for every nonzero homomorphism $\xi : G \to \mathbb{R}$ which vanishes on N. We need to show that $i^! C_*$ is a finitely dominated RN-chain complex. Let $\mathcal{B}$ be an RG basis of C_*. Let $g_1, \ldots, g_k \in G$ be such that their images in $G/N \cong \mathbb{Z}^k$ generate G/N. Then

$$\mathcal{B}_N \quad = \quad \{g_1^{n_1} \cdots g_k^{n_k} x \,|\, x \in \mathcal{B}, n_i \in \mathbb{Z}, i = 1, \ldots, k\}$$

is an RN basis of $i^!C_*$. We want to define a norm on C_* which will behave similarly to the map $|H| : X_N \to [0,\infty)$ used in the proof of Theorem 3.2.

For every $y \in C_*$ define $\operatorname{supp} y$ as a subset of G inductively by starting in dimension 0. Every $y \in C_0$ can be written uniquely as $y = \sum_{x\in\mathcal{B}_0} y_x x$ with $y_x \in RG$ and we let

$$\operatorname{supp} y \quad = \quad \bigcup_{x\in\mathcal{B}_0} \operatorname{supp} y_x.$$

Now for $y \in C_i$ we have $y = \sum_{x\in\mathcal{B}_i} y_x x$ with $y_x \in RG$ and we define

$$\operatorname{supp} y \quad = \quad \bigcup_{x\in\mathcal{B}_i} \operatorname{supp} y_x \cup \operatorname{supp} \partial y.$$

Notice that $\operatorname{supp} y$ is a finite subset of G and depends on the basis.

If $g \in G$, we look at the image of g in $G/N \cong \mathbb{Z}^k$. This includes into $\mathbb{R}^k$ be letting the image of g_i correspond to the standard basis $e_i \in \mathbb{R}^k$. Denote the image of g in $\mathbb{R}^k$ by $e(g)$.

Now we define $\|\cdot\| : C_* \to [0,\infty) \cup \{-\infty\}$ by

$$\|y\| \quad = \quad \begin{cases} -\infty & \text{if } y = 0 \\ \max\{|e(g)|\,|g \in \operatorname{supp} y\} & \text{if } y \neq 0 \end{cases}$$

Here $|e(g)|$ denotes the Euclidean norm of $e(g) \in \mathbb{R}^k$.

The definition of the norm $\|\cdot\|$ is similar to the definition of the norm in Bieri and Renz [1, §5]. In particular we also get that for every $R > 0$ the set $D^R_* = \{y \in i^!C_* \,|\, \|y\| \leq R\}$ is a finitely generated free RN chain complex generated by the elements $u \in \mathcal{B}_N$ which satisfy $\|u\| \leq R$. We prove that $i^!C_*$ is dominated by D^R_* provided $R > 0$ is large enough.

For this we need an analogue of Lemma 3.5. Assume there exist constants $R > 0$, $\varepsilon > 0$, $B > 0$, $C > 0$ and a chain homotopy $K : i^!C_* \to i^!C_{*+1}$ with $K : 1 \simeq a$ such that

$$\varepsilon \;\leq\; \|x\| - \|a(x)\| \;\leq\; C \tag{11}$$

ll $x \in i^!C_*$ with $\|x\| \geq R$ and

$$-B \;\leq\; \|x\| - \|K(x)\| \;\leq\; C \tag{12}$$

for all $x \in i^!C_*$ with $\|x\| \geq R$. We want to show that then $i^!C_*$ is finitely dominated. The proof proceeds as the proof of Lemma 3.5.

Since $K + Ka : 1 \simeq a^2$ we can assume that $\varepsilon \geq 2B$. There is an $L \geq R + B$ such that $\|a(x)\| \leq L$ and $\|K(x)\| \leq L$ for all $x \in D^R_*$. Define $\lambda : i^!C_* \to \{0,1\}$ by $\lambda(x) = 0$ if $\|x\| \leq L$ and $\lambda(x) = 1$ if $\|x\| > L$. Then for every $x \in i^!C_*$ the sequence $(\lambda(a^l(x)))_{l=0}^\infty$ is monotonely decreasing and 1 for only finitely many terms.

We define a chain homotopy $\mathcal{K} : i^!C_* \to i^!C_{*+1}$ by defining it on basis elements $x \in \mathcal{B}_N$ as

$$\mathcal{K}(x) \quad = \quad \sum_{l=0}^{\infty} \lambda(a^l(x))K(a^l(x)).$$

If $x \in (\mathcal{B}_N)_s$ there exist $y_1, \ldots, y_u \in (\mathcal{B}_N)_{s-1}$ and $(\partial x)_1, \ldots, (\partial x)_u \in RN$ such that

$$\partial x = \sum_{j=1}^{u} (\partial x)_j y_j.$$

Then

$$\begin{aligned} \partial\mathcal{K}(x) + \mathcal{K}\partial(x) &= \sum_{l=0}^{\infty} \partial(\lambda(a^l(x))Ka^l(x)) + \sum_{j=1}^{u} (\partial x)_u \sum_{i=0}^{\infty} \lambda(a^l(y_j))Ka^l(y) \\ &= \sum_{i=0}^{\infty} \left(\lambda(a^l(x))\partial Ka^l(x) + \sum_{j=1}^{u} \lambda(a^l(y))Ka^l(\partial x)_j y \right) \end{aligned}$$

Let $m \in \mathbb{Z}$ be an integer such that $\lambda(a^l(x)) = 0 = \lambda(a^l(y_j))$ for all $j = 1, \ldots, u$ and $l > m$. Then there exist $r_1 \in i^! C_s$ and $r_2 \in i^! C_{s-1}$ with $\|r_1\|, \|r_2\| \leq L$ such that

$$\begin{aligned} \partial\mathcal{K}(x) + \mathcal{K}\partial(x) &= \sum_{l=0}^{m} \big(\partial Ka^l(x) + Ka^l\partial(x)\big) + \partial K(r_1) + K(r_2) \\ &= (1 - a^{m+1})(x) + R(x) \end{aligned}$$

with $\|R(x)\| \leq L + C$. Also $\|a^{m+1}(x)\| \leq L$ since $\lambda(a^{m+1}(x)) = 0$ by the choice of m. It follows that $\mathcal{K}$ is a chain homotopy $\mathcal{K} : 1 \simeq b$ with $b : i^! C_* \to i^! C_*$ a chain map with $\|b(x)\| \leq L + C$ for all $x \in \mathcal{B}_N$. Therefore the image of b is contained in D_*^{L+C} and $i^! C_*$ is finitely dominated provided we can find the constants and K as in (11) and (12).

This is done as in the proof of Theorem 3.2. The topological constructions can be transformed into chain complex constructions just as we did above with the constructions from the proof of Lemma 3.5. The details will be left to the reader. □

Let us now return to the situation where X is a finite CW-complex. Then $C_*(\tilde{X})$ is also a free $\mathbb{Z}N$-complex, but for $k \geq 1$ it is not finitely generated. As a $\mathbb{Z}N$-complex we write it as $C_*(X_N; \mathbb{Z}N)$. By Wall [19], see also Hughes and Ranicki [7, Th.6.8], X_N is finitely dominated if and only if N is finitely presented and $C_*(X_N; \mathbb{Z}N)$ is homotopy equivalent to a finitely generated projective $\mathbb{Z}N$-complex D_*. This leads to the following extension of Theorem 3.2.

Theorem 4.9. *Let X be a finite CW-complex and $N \leq \pi_1(X)$ a normal subgroup such that $\pi_1(X)/N \cong \mathbb{Z}^k$ for some $k \geq 0$. Then the following are equivalent.*

1. *X_N is finitely dominated.*
2. *$S(\pi_1(X); N) \subset \Sigma(X)$.*
3. *$S(\pi_1(X); N) \subset \Sigma(X; \mathbb{Z})$ and N is finitely presented.* □

Recall that $S(\pi_1(X); N)$ are equivalence classes of nonzero homomorphisms $\xi : \pi_1(X) \to \mathbb{R}$ vanishing on N.

Remark 4.10. An alternative proof of (2) ⇔ (3) can be given as follows: by Latour [8, Cor.5.23] X is ξ-contractible if and only if ξ is stable and $C_*(X; \widehat{\mathbb{Z}G_\xi})$ is acyclic. See Latour [8, §5] for the definition of ξ being stable. But ξ is stable for every $\xi \in S(\pi_1(X); N)$ if and only if N is finitely presented by Bieri and Renz [1, Rm.6.5]. Thus we get a proof of Theorem 4.9 independent of Wall [19]. Of course the proof involving Theorem 4.7 is independent of Latour [8, §5].

5. Obstructions for fibering and nonsingular closed 1-forms

Let X be a finite connected CW-complex and $\xi : \pi_1(X) \to \mathbb{Z}$ a surjective homomorphism. Let $\overline{X}$ be the covering space corresponding to $\ker\xi$. Then ξ factors through $\Delta(\overline{X} : X) \cong \pi_1(X)/\ker\xi$ and there is a covering transformation $t : \overline{X} \to \overline{X}$ with $\xi(t) = 1$. Assume that $\overline{X}$ is finitely dominated and let K be a finite CW-complex, $a : K \to \overline{X}$, $b : \overline{X} \to K$ be cellular maps such that $ab \simeq \mathrm{id}_{\overline{X}}$. By combining Lemma 2.10 with Proposition 2.8 we have

$$X \quad \simeq \quad T(bta : K \to K)$$

and let $h : T_{bta} \to X$ be the resulting homotopy equivalence. Both X and T_{bta} are finite CW-complexes so we have a Whitehead torsion

$$\tau(h) \quad \in \quad \mathrm{Wh}(\pi_1(X))$$

Because of Proposition 2.8 $\tau(h)$ does not depend on the choice of the finite domination of $\overline{X}$. Notice that replacing t by t^{-1} gives another homotopy equivalence $T_{bt^{-1}a} \to X$. We put this together in the following definition.

Definition 5.1. Let X be a finite connected CW-complex, $\xi : \pi_1(X) \to \mathbb{Z}$ a surjective homomorphism such that the covering space $\overline{X}$ corresponding to $\ker\xi$ is finitely dominated. Then the *fibering obstructions* $\Phi^+(X,\xi)$ and $\Phi^-(X,\xi)$ are defined as

$$\begin{aligned}\Phi^+(X,\xi) &= \tau(h^+) \in \mathrm{Wh}(\pi_1(X))\\ \Phi^-(X,\xi) &= \tau(h^-) \in \mathrm{Wh}(\pi_1(X))\end{aligned}$$

where $h^+ : T_{bta} \to X$ and $h^- : T_{bt^{-1}a} \to X$ are the homotopy equivalences described above.

In the case that M is a closed connected smooth manifold of dimension n we get

$$\Phi^+(M,\xi) \quad = \quad (-1)^{n-1}(\Phi^-(M,\xi))^*$$

where $^* : \mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}(\pi_1(M))$ is induced by the orientation involution of $\mathbb{Z}\pi_1(M)$, see Hughes and Ranicki [7, Rm.15.12]. In general the vanishing of $\Phi^+(X,\xi)$ does not imply the vanishing of $\Phi^-(X,\xi)$; see Hughes and Ranicki [7, §15] for relations between Φ^+ and Φ^-.

The Farrell obstruction $\tau_F(M,f)$ which appears in Theorem 1.1 is given by

$$\tau_F(M,f) \quad = \quad \Phi^+(M,f_\#).$$

This explains the name 'fibering obstruction' in Definition 5.1.

Recall that the infinite cyclic covering space $\overline{X}$ corresponding to the kernel of $\xi : \pi_1(X) \to \mathbb{Z}$ is finitely dominated if and only if X is $(\pm\xi)$-contractible. But if X is ξ-contractible for any $\xi : \pi_1(X) \to \mathbb{R}$ we have already defined a torsion given by

$$\tau(X,\xi) \quad = \quad \tau(C_*(X;\widehat{\mathbb{Z}\pi_1(X)}_\xi)) \in \mathrm{Wh}(\pi_1(X);\xi)$$

It turns out that $\Phi^+(X,\xi)$ determines $\tau(X,\xi)$.

Proposition 5.2. *Let X be a finite connected CW-complex, $\xi : \pi_1(X) \to \mathbb{Z}$ a surjective homomorphism such that the covering space corresponding to $\ker\xi$ is finitely dominated. Then*

$$i_*\Phi^+(X,\xi) \quad = \quad \tau(X,\xi)$$

where $i_* : \mathrm{Wh}(\pi_1(X)) \to \mathrm{Wh}(\pi_1(X);\xi)$ *is the natural homomorphism induced by the inclusion* $\mathbb{Z}\pi_1(X) \to \widehat{\mathbb{Z}\pi_1(X)}_\xi$.

A proof can be found in Ranicki [14, Prop.15.15] which decomposes the various torsions into the components of the Bass-Heller-Swan decomposition, thus obtaining further information about the relation to the finiteness obstruction of Wall. I am indebted to Andrew Ranicki for pointing out the following, more elementary proof of Proposition 5.2 which will remain useful later. First we need a result on the torsion of a mapping torus, compare Geoghegan and Nicas [6, Th.7.6].

Lemma 5.3. *Let* X *be a finite connected CW-complex and* $f : X \to X$ *a cellular map. Let* $g : T_f \to S^1$ *be the canonical projection. Then*

$$\tau(T_f, g_\#) \quad = \quad 0 \in \mathrm{Wh}(\pi_1(T_f); g_\#).$$

Proof. Notice that T_f is $g_\#$-contractible by Lemma 2.9 so $\tau(T_f, g_\#)$ is defined. Let $G = \pi_1(T_f)$ and $\tilde{T}_f$ be the universal covering space of T_f. Note that we have covering spaces $\tilde{T}_f \to \overline{T}_f \to T_f$ and a natural control map $h : \overline{T} \to \mathbb{R}$ which gives a natural map $\tilde{h} : \tilde{T} \to \mathbb{R}$.

Now

$$C_*(T_f; \mathbb{Z}G) \quad = \quad C_*(X; \mathbb{Z}G) \oplus C_{*-1}(X; \mathbb{Z}G)$$

and we can choose a basis of $C_*(T_f; \mathbb{Z}G)$ by choosing lifts of cells of X in $\tilde{h}^{-1}(\{1\})$ and lifts of cells of the form $\sigma \times (0,1)$ in $\tilde{h}([0,1])$. With respect to such a basis the matrix of the boundary operator in degree i looks like

$$\partial_i \quad = \quad \begin{pmatrix} \partial_i^X & (-1)^{i-1}(I - A_i t) \\ 0 & \partial_{i-1}^X \end{pmatrix}$$

where A_i is a matrix over $\mathbb{Z}H$ with $H = \ker g_\#$ and $t \in G$ satisfies $g_\#(t) = -1$. Thus $C_*(T_f; \mathbb{Z}G)$ can be thought of as the mapping cone $\mathcal{C}(\varphi)$ of a $\mathbb{Z}G$-chain map $\varphi = \mathrm{id} - at : C_*(X; \mathbb{Z}G) \to C_*(X; \mathbb{Z}G)$. Notice that a is induced by $f : X \to X$. Also

$$\mathrm{id}_{\widehat{\mathbb{Z}G}_\xi} \otimes \varphi : C_*(X; \widehat{\mathbb{Z}G}_\xi) \to C_*(X; \widehat{\mathbb{Z}G}_\xi)$$

is an automorphism in every degree. Now

$$C_*(T_f; \widehat{\mathbb{Z}G}_\xi) \quad = \quad \mathcal{C}(\mathrm{id}_{\widehat{\mathbb{Z}G}_\xi} \otimes \varphi)$$

and

$$\tau(\mathcal{C}(\mathrm{id} \otimes \varphi)) \quad = \quad \sum_{i=0}^{\infty} (-1)^i \tau(I - A_i t) \in \mathrm{Wh}(G; g_\#)$$

by Ranicki [15, Prop.1.7(ii)]. But obviously $\tau(I - A_i t) = 0 \in \mathrm{Wh}(G; g_\#)$ so the result follows. □

Proof of Proposition 5.2. Let $G = \pi_1(X)$. The homotopy equivalence $h : T_{bta} \to X$ induces a short exact sequence of chain complexes

$$0 \longrightarrow C_*(X; \mathbb{Z}G) \longrightarrow \mathcal{C}(h) \longrightarrow C_{*-1}(T_{bta}; \mathbb{Z}G) \longrightarrow 0$$

where only $\mathcal{C}(h)$ is acyclic. After tensoring with $\widehat{\mathbb{Z}G}_\xi$ we get a short exact sequence of acyclic complexes

$$0 \longrightarrow C_*(X; \widehat{\mathbb{Z}G}_\xi) \longrightarrow \mathcal{C}(\mathrm{id} \otimes h) \longrightarrow C_{*-1}(T_{bta}; \widehat{\mathbb{Z}G}_\xi) \longrightarrow 0$$

and so

$$\tau(\mathrm{id} \otimes h) \quad = \quad \tau(X, \xi) - \tau(T_{bta}, \xi).$$

But $\tau(T_{bta}, \xi) = 0$ by Lemma 5.3 and obviously $i_*\Phi^+(X, \xi) = \tau(\mathrm{id} \otimes h)$. □

Definition 5.4. A nonzero homomorphism $\xi : G \to \mathbb{R}$ is called *rational*, if $\operatorname{im} \xi$ is infinite cyclic. Otherwise it is called *irrational*.

If $\xi : \pi_1(X) \to \mathbb{R}$ is a rational homomorphism such that X is $(\pm\xi)$-contractible, then we have the fibering obstructions $\Phi^+(X, \xi)$, $\Phi^-(X, \xi) \in \mathrm{Wh}(G)$ such that

$$\begin{aligned} i_*\Phi^+(X, \xi) &= \tau(X, \xi) \in \mathrm{Wh}(\pi_1(X); \xi) \\ i_*\Phi^-(X, \xi) &= \tau(X, -\xi) \in \mathrm{Wh}(\pi_1(X); -\xi) \end{aligned}$$

For an irrational $\xi : \pi_1(X) \to \mathbb{R}$ it is not clear how to define an element $\Phi(X, \xi) \in \mathrm{Wh}(\pi_1(X))$ such that $i_*\Phi(X, \xi) = \tau(X, \xi) \in \mathrm{Wh}(\pi_1(X); \xi)$. It is not even known in general whether the natural map $i_* : \mathrm{Wh}(G) \to \mathrm{Wh}(G; \xi)$ is surjective. In the case of a rational homomorphism this follows easily from Pajitnov and Ranicki [11, Mn.Th.] which also shows that i_* need not be injective.

But it turns out that in the case of an $\mathbb{Z}^k$-covering space X_N which is finitely dominated with $k \geq 2$ the fibering obstructions are the same for all rational homomorphisms $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$ and that any such obstruction determines $\tau(X, \xi)$ even for irrational $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$.

Proposition 5.5. *Let X be a finite connected CW-complex, $N \leq \pi_1(X)$ a normal subgroup such that $\pi_1(X)/N \cong \mathbb{Z}^k$ for an integer $k \geq 2$ and X_N is finitely dominated. Let $\xi : \pi_1(X) \to \mathbb{R}$ be a nonzero rational homomorphism with $N \leq \ker \xi$.*

(1) *For every nonzero homomorphism $\xi' : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi'$ we have*

$$i_*\Phi^+(X, \xi) \quad = \quad \tau(X, \xi') \in \mathrm{Wh}(\pi_1(X); \xi').$$

(2) *For every nonzero rational homomorphism $\xi' : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi'$ we have*

$$\Phi^+(X, \xi) \quad = \quad \Phi^+(X, \xi') \in \mathrm{Wh}(\pi_1(X)).$$

Remark 5.6. Part (2) is an unpublished result of Farrell. It shows in particular that $\Phi^-(X, \xi) = \Phi^+(X, \xi)$ in the situation of Proposition 5.5.

Proof of Proposition 5.5. Let us start with the case that $\pi_1(X)/N \cong \mathbb{Z}^2$. We can assume that ξ is a surjective homomorphism $\xi : \pi_1(X) \to \mathbb{Z}$. Abbreviate $G = \pi_1(X)$ and $H = \ker \xi$. We have two infinite cyclic covering spaces

$$X_N \longrightarrow X_H \longrightarrow X.$$

Let $t_2 : X_N \to X_N$ be a generator of $\Delta(X_N : X_H) \cong \mathbb{Z}$. Notice that t_2 also represents an element $\Delta(X_N : X)$ and $\xi(t_2) = 0$. Also let $t_1 : X_H \to X_H$ be a generator of $\Delta(X_H : X) \cong \mathbb{Z}$ with $\xi(t_1) = 1$.

Let K be a finite CW-complex such that cellular maps $a : K \to X_N$ and $b : X_N \to K$ exist with $ab \simeq \mathrm{id}_{X_N}$ By Proposition 2.8 and Lemma 2.10 we have

$$X_H \quad \simeq \quad T(bt_2a : K \to K). \tag{13}$$

Let $c : T_{bt_2a} \to X_H$ and $d : X_H \to T_{bt_2a}$ denote the homotopy equivalences given by (13).

Now let $\xi_1 : G \to \mathbb{R}$ be a nonzero rational homomorphism with $N \leq \ker \xi_1$ and

$\xi_1(t_2) > 0$. Then ξ_1 restricts to a nonzero rational homomorphism $\xi_1| : H \to \mathbb{R}$ and $C_*(T_{bt_2a}; \widehat{\mathbb{Z}H}_{\xi_1|})$ is acyclic by Lemma 5.3. Again by Proposition 2.8 and Lemma 2.10 we have

$$X \quad \simeq \quad T(dt_1c : T_{bt_2a} \to T_{bt_2a})$$

and the torsion of the homotopy equivalence $h : T_{dt_1c} \to X$ represents $\Phi^+(X, \xi)$. The chain complex $C_*(T_{dt_1c}; \mathbb{Z}G)$ fits into a short exact sequence of $\mathbb{Z}G$-chain complexes

$$0 \longrightarrow C_*(T_{bt_2a}; \mathbb{Z}G) \longrightarrow C_*(T_{dt_1c}; \mathbb{Z}G) \longrightarrow C_{*-1}(T_{bt_2a}; \mathbb{Z}G) \longrightarrow 0$$

where $C_*(T_{bt_2a}; \mathbb{Z}G) = \mathbb{Z}G \otimes_{\mathbb{Z}H} C_*(T_{bt_2a}; \mathbb{Z}H)$.

We have an inclusion of rings

$$\widehat{\mathbb{Z}H}_{\xi_1|} \longrightarrow \widehat{\mathbb{Z}G}_{\xi_1 + C\cdot\xi}$$

for every $C \in \mathbb{R}$. Write $\xi_C = \xi_1 + C \cdot \xi$. Then we have that

$$\begin{aligned} C_*(T_{bt_2a}; \widehat{\mathbb{Z}G}_{\xi_C}) &= \widehat{\mathbb{Z}G}_{\xi_C} \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}H} C_*(T_{bt_2a}; \mathbb{Z}H) \\ &= \widehat{\mathbb{Z}G}_{\xi_C} \otimes_{\widehat{\mathbb{Z}H}_{\xi_1|}} \widehat{\mathbb{Z}H}_{\xi_1|} \otimes_{\mathbb{Z}H} C_*(T_{bt_2a}; \mathbb{Z}H) \end{aligned}$$

is acyclic for every $C \in \mathbb{R}$.

Therefore we have a short exact sequence of acyclic chain complexes

$$0 \longrightarrow C_*(T_{bt_2a}; \widehat{\mathbb{Z}G}_{\xi_C}) \longrightarrow C_*(T_{dt_1c}; \widehat{\mathbb{Z}G}_{\xi_C}) \longrightarrow C_{*-1}(T_{bt_2a}; \widehat{\mathbb{Z}G}_{\xi_C}) \longrightarrow 0$$

for every $C \in \mathbb{R}$. Therefore

$$\tau(T_{dt_1c}, \xi_C) \quad = \quad 0 \in \mathrm{Wh}(G; \xi_C)$$

for every $C \in \mathbb{R}$ and hence

$$\begin{aligned} i_*\Phi^+(X, \xi) &= \tau(X, \xi_C) - \tau(T_{dt_1c}, \xi_C) \\ &= \tau(X, \xi_C) \in \mathrm{Wh}(G; \xi_C) \end{aligned}$$

for every $C \in \mathbb{R}$.

Replacing ξ_1 by $-\xi_1$ and t_2 by t_2^{-1} shows that

$$i_*\Phi^+(X, \xi) \quad = \quad \tau(X, \pm\xi_C) \in \mathrm{Wh}(G; \pm\xi_C) \tag{14}$$

for all $C \in \mathbb{R}$. Notice that the definition of $\Phi^+(X, \xi)$ does not depend on the choice of finite domination so we can replace T_{bt_2a} by $T_{bt_2^{-1}a}$.

Furthermore by switching ξ_1 and ξ we get

$$i_*\Phi^+(X, \xi_1) \quad = \quad \tau(X, \pm(\xi + C \cdot \xi_2)) \in \mathrm{Wh}(G; \pm(\xi + C \cdot \xi_2)) \tag{15}$$

for all $C \in \mathbb{R}$. In particular for $C \neq 0$ we get that $\Phi^+(X, \xi) - \Phi^+(X, \xi_1)$ is in the kernel of

$$(i_*, i_*) : \mathrm{Wh}(G) \longrightarrow \mathrm{Wh}(G; \xi + C \cdot \xi_1) \oplus \mathrm{Wh}(G; -(\xi + C \cdot \xi_1)).$$

By choosing $C \neq 0$ appropriately we can assume that $\xi + C \cdot \xi_1$ is rational in which case (i_*, i_*) is injective by Pajitnov and Ranicki [11, Mn.Th.]. This finishes the proof of (2).

Now let $\xi_1 : G \to \mathbb{R}$ be a rational homomorphism with $\xi_1(t_1) = 0$. Then for any nonzero homomorphism $\xi' : G \to \mathbb{R}$ with $N \leq \ker \xi'$ we can find $a, b \in \mathbb{R}$ with $\xi' = a \cdot \xi + b \cdot \xi_1$ and (1) follows in the case $k = 2$ either from (14) or from (15).

So now assume that $k > 2$. Notice that we only have to show (1) since (2) always follows from the case $k = 2$. To see this note that there is always a $\mathbb{Z}^2$-covering space $X_H \to X$ such that $N \le H \le \ker \xi \cap \ker \xi'$ and $H/N \cong \mathbb{Z}^{k-2}$. But then X_H is homotopy finite by Corollary 2.11.

To prove (1) we need another Lemma.

Lemma 5.7. *Let T be a finite connected CW-complex, $N \le \pi_1(T)$ a normal subgroup with $\pi_1(T)/N \cong \mathbb{Z}^k$ with $k \ge 2$. Assume that T_N is finitely dominated. Let $\xi_1, \xi_2 : \pi_1(T) \to \mathbb{Z}$ be the projections to the first two summands in $\pi_1(T)/N \cong \mathbb{Z}^k \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^{k-2}$. Let $H = \ker \xi_1$.*

Then there exist finite CW-complexes T^+, T^- homotopy equivalent to T_H such that

$$C_*(T^+; \widehat{\mathbb{Z}\pi_1(T)}_\chi) \text{ and } C_*(T^-; \widehat{\mathbb{Z}\pi_1(T)}_{-\chi})$$

are acyclic for every homomorphism $\chi = \xi_2 + \eta : \pi_1(T) \to \mathbb{R}$ where $\eta(g) = 0$ for every $g \in N$ and every $g \in \pi_1(T)$ with $\xi_2(g) \neq 0$.

Proof. The proof is by induction, for $k = 2$ this was proven above with $T^+ = T_{bt_2a}$ and $T^- = T_{bt_2^{-1}a}$.

So assume that $k > 2$. By Corollary 2.11 we have that T_H is homotopy equivalent to a finite CW-complex Y. Let us identify $\pi_1(Y)$ with $\pi_1(T_H)$ so that $N \le \pi_1(Y)$ is a normal subgroup with $\pi_1(Y)/N \cong \mathbb{Z}^{k-1}$ and $k - 1 \ge 2$. Now ξ_2 restricts to a surjective homomorphism $\xi_2|\pi_1(Y) \to \mathbb{Z}$. Let $K = \ker \xi_2|\pi_1(Y)$. By induction hypothesis there exist finite CW-complexes F^+, F^- homotopy equivalent to Y_K such that

$$C_*(F^+; \widehat{\mathbb{Z}\pi_1(Y)}_{\bar\chi}) \text{ and } C_*(F^-; \widehat{\mathbb{Z}\pi_1(Y)}_{-\bar\chi})$$

are acyclic for every homomorphism $\bar\chi = \xi_2| + \eta : \pi_1(Y) \to \mathbb{R}$ where η vanishes on N and $\eta(g) = 0$ for every $g \in \pi_1(Y)$ with $\xi_2(g) \neq 0$.

Now if $\chi : \pi_1(T) \to \mathbb{R}$ is of the form $\chi = \xi_2 + \eta$ with $\eta|N = 0$ and $\eta(g) = 0$ for every $g \in \pi_1(T)$ with $\xi_2(g) \neq 0$, we have inclusions of Novikov rings

$$\begin{array}{ccc} \widehat{\mathbb{Z}\pi_1(Y)}_{\xi_2|+\eta|} & \longrightarrow & \widehat{\mathbb{Z}\pi_1(T)}_{\xi_2+\eta} \\ \widehat{\mathbb{Z}\pi_1(Y)}_{-(\xi_2|+\eta|)} & \longrightarrow & \widehat{\mathbb{Z}\pi_1(T)}_{-(\xi_2+\eta)} \end{array}$$

Denote $a^\pm : F^\pm \to Y_K$, $b^\pm : Y_K \to F^\pm$ the homotopy equivalences and $t : Y_K \to Y_K$ the covering transformation with $\xi_2(t) = 1$. Let

$$\begin{array}{rcl} T^+ & = & T(b^+ta^+ : F^+ \to F^+) \\ T^- & = & T(b^-t^{-1}a^- : F^- \to F^-) \end{array}$$

be finite CW-complexes homotopy equivalent to Y, hence also to T_H. There are short exact sequences of chain complexes

$$0 \longrightarrow C_*(F^+; \mathbb{Z}\pi_1(Y)) \longrightarrow C_*(T^+; \mathbb{Z}\pi_1(Y)) \longrightarrow C_{*-1}(F^+; \mathbb{Z}\pi_1(Y)) \longrightarrow 0$$
$$0 \longrightarrow C_*(F^-; \mathbb{Z}\pi_1(Y)) \longrightarrow C_*(T^-; \mathbb{Z}\pi_1(Y)) \longrightarrow C_{*-1}(F^-; \mathbb{Z}\pi_1(Y)) \longrightarrow 0$$

It follows that $C_*(T^+; \widehat{\mathbb{Z}\pi_1(Y)}_{\chi|})$ and $C_*(T^-; \widehat{\mathbb{Z}\pi_1(Y)}_{-\chi|})$ are acyclic and therefore the same is true after tensoring with $\widehat{\mathbb{Z}\pi_1(T)}_{\pm\chi}$. This finishes the proof of the lemma. $\square$

Let us return to the proof of Proposition 5.5. Recall that we assumed ξ to be a surjective homomorphism $\xi : G \to \mathbb{Z}$ and it factors as $G \to G/N \cong \mathbb{Z}^k \to \mathbb{Z}$. So we can think of ξ as the projection to the first factor of $\mathbb{Z}^k \cong G/N$. Let $\xi_2 : G \to \mathbb{Z}$ be projection to the second summand. Recall that $H = \ker \xi$. We can apply Lemma 5.7 so there exist $T^{\pm}$ homotopy equivalent to X_H with the acyclicity property described there. Let $\xi' : G \to \mathbb{R}$ be any nonzero homomorphism which vanishes on N. Then there exist $a, b \in \mathbb{R}$ and a homomorphism $\eta : G \to \mathbb{R}$ which vanishes on N and for every $g \in G$ with $\xi_2(g) \neq 0$ such that $\xi' = a \cdot \xi_2 + b \cdot \eta$.

If $a = 0$, then $G/\ker \xi' \cong \mathbb{Z}^l$ with $l < k$ and we get $i_*\Phi^+(X, \xi) = \tau(X, \xi')$ in the case $l \geq 2$ by induction or in the case $l = 1$ by Proposition 5.2 using (2).

So without loss of generality assume $a = 1$. Now $X \simeq T(bta : T^+ \to T^+)$ where the maps b, t and a are as before. Because of Lemma 5.7 and the usual exact sequence of acyclic complexes we get

$$\tau(C_*(T_{bta}; \widehat{\mathbb{Z}G}_{\xi'})) \quad = \quad 0 \; \in \; \mathrm{Wh}(G; \xi')$$

and therefore

$$i_*\Phi^+(X, \xi) \quad = \quad \tau(X, \xi') \; \in \; \mathrm{Wh}(G; \xi')$$

which is what we had to show. □

Corollary 5.8. *Let X be a finite connected CW-complex, $N \leq \pi_1(X)$ a normal subgroup such that $\pi_1(X)/N \cong \mathbb{Z}^k$ for an integer $k \geq 2$ and X_N is finitely dominated. Then the following are equivalent.*

1. *There is a nonzero rational homomorphism $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$ with $\Phi^+(X, \xi) = 0$.*
2. *For all nonzero rational homomorphisms $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$ we have $\Phi^+(X, \xi) = 0$.*
3. *For all nonzero homomorphisms $\xi : \pi_1(X) \to \mathbb{R}$ with $N \leq \ker \xi$ we have $\tau(X, \xi) = 0$.* □

Remark 5.9. Given a nonzero rational homomorphism $\xi : \pi_1(X) \to \mathbb{R}$ we can find a positive real number c such that $c \cdot \xi : \pi_1(X) \to \mathbb{Z}$ is surjective. Then there exists a map $f : M \to S^1$ such that $f_\# : \pi_1(X) \to \pi_1(S^1) \cong \mathbb{Z}$ is $c \cdot \xi$. Assume that the covering space corresponding to $\ker \xi$ is finitely dominated. By Ranicki [14, Prop.2.7] we have that $\Phi^+(X, \xi) = 0 = \Phi^-(X, \xi)$ is equivalent to the existence of a simple homotopy equivalence $h : X \to T(k : K \to K)$ to the mapping torus of a simple homotopy equivalence $k : K \to K$, where K is a finite connected CW-complex, such that $f \simeq gh$. Here $g : T_k \to S^1$ is the canonical projection. Furthermore $\Phi^+(X, \xi) = 0 = \Phi^-(X, \xi)$ is equivalent to $\tau(X, \xi) = 0 = \tau(X, -\xi)$. It would be interesting to have a similar geometric condition equivalent to the vanishing of both $\tau(X, \xi)$ and $\tau(X, -\xi)$ for an irrational homomorphism ξ. Of course in the case of a manifold such an interpretation is given by Latour's Theorem.

Let us apply Corollary 5.8 to Latour's Theorem 1.2.

Theorem 5.10. *Let M be a closed connected smooth manifold with $\dim M \geq 6$, $N \leq \pi_1(M)$ a normal subgroup such that $\pi_1(M)/N \cong \mathbb{Z}^k$ for some $k \geq 1$ and M_N is finitely dominated. Then the following are equivalent.*

1. *There is a nonzero $\xi : \pi_1(M) \to \mathbb{R}$ with $N \leq \ker \xi$ which can be represented by a nonsingular closed 1-form.*
2. *Every nonzero $\xi : \pi_1(M) \to \mathbb{R}$ with $N \leq \ker \xi$ can be represented by a nonsingular closed 1-form.*

Proof. If $k = 1$, up to multiplication by a positive real number there are only two nonzero homomorphisms which vanish on N. If ω is a nonsingular closed 1-form representing one such homomorphism, then $-\omega$ is a nonsingular closed 1-form representing the other.

If $k \geq 2$ and there is a nonsingular closed 1-form representing some $\xi : \pi_1(M) \to \mathbb{R}$, then $\tau(M, \xi) = 0$ by Theorem 1.2. Rational homomorphisms are dense in $S(\pi_1(M))$ and in $S(\pi_1(M); N)$ so by Proposition 4.3 there is a rational homomorphism $\xi' : \pi_1(M) \to \mathbb{R}$ near ξ which vanishes on N and we have $\tau(M, \xi') = 0$. This can also be derived directly by the geometric argument of Tischler [18]. Combining Theorems 1.1 and 1.2 we get $\Phi^+(M, \xi') = 0$. By Corollary 5.8 we get $\tau(M, \xi'') = 0$ for all $\xi'' \in S(\pi_1(M); N)$. Therefore (2) follows from Theorem 1.2. □

We want to finish with two examples which show that the finiteness properties of X_N do not have an immediate impact on the existence of nonsingular closed 1-forms.

Example 5.11. Let M be a closed connected smooth manifold which has an infinite cyclic covering space $\overline{M}$ corresponding to a rational homomorphism $\xi : \pi_1(M) \to \mathbb{R}$ that is finitely dominated but not homotopy finite. In particular there is no nonsingular closed 1-form representing ξ. To see that such manifolds exist, let H be a finitely presented group and $x \in \tilde{K}_0(\mathbb{Z}H)$. By the Existence Theorem of Siebenmann [16, Thm.8.6] there exists for every $n \geq 5$ a smooth manifold W with $n = \dim W$, compact boundary and one tame end ε such that $\pi_1(\varepsilon) \cong H$ and the Siebenmann end obstruction is

$$\sigma(\varepsilon) \quad = \quad x \ \in \ \tilde{K}_0(\mathbb{Z}H).$$

Let us assume that $n \geq 6$. Then by Hughes and Ranicki [7, Thm.19] we can find a closed connected smooth manifold M with $\dim M = n$ such that $\pi_1(M) \cong H \times \mathbb{Z}$ and

$$\Phi^+(M, \xi) \quad = \quad x \ \in \ \tilde{K}_0(\mathbb{Z}H) \ \leq \ \mathrm{Wh}(H \times \mathbb{Z}).$$

Here $\xi : H \times \mathbb{Z} \to \mathbb{Z}$ is projection onto $\mathbb{Z}$. It follows from Ranicki [14, Prop.15.15] that the Wall finiteness obstruction of $\overline{M}$ is

$$[\overline{M}] \quad = \quad (-1)^n x^* - x \ \in \ \tilde{K}_0(\mathbb{Z}H)$$

If H is a finite group of odd order, $^* : \tilde{K}_0(\mathbb{Z}H) \to \tilde{K}_0(\mathbb{Z}H)$ is the standard involution and if $2x \neq 0 \in \tilde{K}_0(\mathbb{Z}H)$ we cannot have that $x^* = x$ and $x^* = -x$. So if $\tilde{K}_0(\mathbb{Z}H)$ has elements of order bigger than 2 we can find M with $\pi_1(M) \cong H \times \mathbb{Z}$ such that M has a finitely dominated infinite cyclic covering space which is not homotopy finite. See Milnor [10, App.1] or Siebenmann [16, App.] that H with the required properties exists.

Now let $X = M \times S^1$. Then $H \leq \pi_1(X)$ with $\pi_1(X)/H \cong \mathbb{Z}^2$ and $X_H = \overline{M} \times \mathbb{R}$ is finitely dominated, but not homotopy finite. If $\xi' : \pi_1(X) \cong \pi_1(M) \times \mathbb{Z} \to \mathbb{Z}$ is projection to the $\mathbb{Z}$-factor, it is clear that ξ' can be represented by a nonsingular closed 1-form. Hence by Theorem 5.10 every nonzero homomorphism $\xi : \pi_1(X) \to \mathbb{R}$ which vanishes on H can be represented by a nonsingular closed 1-form.

Example 5.12. Let N be a closed connected smooth manifold with $n = \dim N \geq 4$. Let $(W; N \times S^1 \times S^1, M)$ be an h-cobordism such that

$$\tau(W, N \times S^1 \times S^1) + (-1)^{n-1} \bar{\tau}(W, N \times S^1 \times S^1) \quad \neq \quad 0 \ \in \ \mathrm{Wh}(\pi_1(W)).$$

Such h-cobordisms exist, see Milnor [10, §11]. Notice that W gives a homotopy equivalence between closed connected smooth manifolds $h : N \times S^1 \times S^1 \to M$ with

$$\tau(h) \quad = \quad \tau(W, N \times S^1 \times S^1) + (-1)^{n-1}\bar{\tau}(W, N \times S^1 \times S^1). \tag{16}$$

Let H be the image of $\pi_1(N) \leq \pi_1(N \times S^1 \times S^1)$ under $h_\# : \pi_1(N \times S^1 \times S^1) \to \pi_1(M)$. Obviously H is a normal subgroup of $\pi_1(M)$ with $\pi_1(M)/H \cong \mathbb{Z}^2$. Also M_H is homotopically equivalent to N so it is homotopy finite. Let $\xi : \pi_1(M) \to \mathbb{R}$ be a nonzero rational homomorphism which vanishes on H. Then $h^*\xi : \pi_1(N \times S^1 \times S^1) \to \mathbb{R}$ is a nonzero rational homomorphism which vanishes on $\pi_1(N)$. It follows that both $\Phi^+(M, \xi)$ and $\Phi^+(N \times S^1 \times S^1, h^*\xi)$ are defined. From the definition of the fibering obstruction we get that

$$\Phi^+(M, \xi) \quad = \quad h_*\Phi^+(N \times S^1 \times S^1, h^*\xi) + \tau(h) \;\in\; \mathrm{Wh}(\pi_1(M)).$$

But obviously $\Phi^+(N \times S^1 \times S^1, h^*\xi) = 0$ and therefore $\Phi^+(M, \xi) \neq 0$ by (16). It follows from Theorem 5.10 that no homomorphism $\xi : \pi_1(M) \to \mathbb{R}$ which vanishes on H can be represented by a nonsingular closed 1-form. In particular there exists an irrational homomorphism $\xi : \pi_1(M) \to \mathbb{R}$ such that M is $(\pm\xi)$-contractible, but every closed 1-form representing ξ has singularities.

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