ZETA FUNCTIONS FOR GRADIENTS OF CLOSED 1-FORMS

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ABSTRACT. Given a cohomology class $\xi \in H^1(M;\mathbb{R})$ on the closed connected smooth manifold M we look at vector fields v which are gradient-like with respect to ξ, i.e. they admit a Lyapunov form ω, a closed 1-form representing ξ which evaluates the vector field positively whenever $v \neq 0$. Assuming that the set of zeros of v is not too complicated and v does not admit homoclinic cycles, we define a zeta function of v, an algebraic object carrying information about the closed orbit structure of v. We show that this zeta function depends continuously on v in a reasonable sense and discuss relations to chain homotopy equivalences between Novikov complexes.

1. Introduction

Let M be a closed connected smooth manifold. By a gradient-like vector field with respect to a cohomology class $\xi \in H^1(M;\mathbb{R})$ we mean a vector field v such that there exists a closed 1-form ω representing ξ with $\omega = 0$ if and only if $v = 0$ and $\omega(v) > 0$ if $v \neq 0$. This is for example the case if v is dual to ω with respect to a Riemannian metric on M. The condition that $\omega(v) > 0$ has interesting consequences for closed orbits of v: integrating ω along a closed orbit gives a positive number. Now ξ represents a homomorphism $\pi_1(M) \to \mathbb{R}$ which can be described by integrating ω along smooth representatives of elements in the fundamental group. This means that closed orbits represent loops which have positive image under ξ. To be precise closed orbits do not represent elements of $\pi_1(M)$, but only conjugacy classes for which ξ is defined as well. Furthermore one expects closed orbits of longer period to have asymptotically a larger image under ξ. This is certainly the case if ω is nonzero everywhere. Fried [3] used this property to collect information on the closed orbit structure in a power series, the zeta function of the vector field, and showed that it agreed with a certain Reidemeister torsion, thus showing that the zeta function is determined by the topology of M in that case.

That the definition of a zeta function still makes sense for singular flows was first shown by Hutchings and Lee [8] and Pajitnov [11]. They considered the case where v is gradient to a circle valued Morse function. The case of gradients of Morse closed 1-forms was also considered by Hutchings [7]. To get a well defined zeta function, they require a generic transversality condition on v. In this case the topology of M does not determine the zeta function. A correction term coming from the so called Novikov complex enters the picture. This is a chain complex generated by the zeros of ω whose boundary is determined by the trajectories of v between zeros.

It turns out that the main reason for the well definedness of the zeta function comes

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from the nonexistence of homoclinic cycles in the case of transverse gradients. This is the point of view of the present paper. We show that we can define zeta functions for gradient-like vector fields without homoclinic cycles provided their singularities are not too pathological. In particular we allow certain degeneracies. The main theorem we obtain is

Theorem 1.1. *Let $\xi : G \to \mathbb{R}$ be a homomorphism where G is the fundamental group of the closed connected smooth manifold M. Then $\zeta : \mathcal{G}(\xi) \to \widehat{HH}_1(\mathbb{Z}G)_\xi$ sending v to $\zeta(-v)$ is continuous.*

Here $\mathcal{G}(\xi)$ is the space of gradient-like vector fields for which we define zeta functions with the C^0-topology. The object $\widehat{HH}_1(\mathbb{Z}G)_\xi$ is a completion of $HH_1(\mathbb{Z}G)$, the first Hochschild homology group of the group ring, and as such it carries a natural topology. Here G is the fundamental group of M and $\zeta(-v)$ the zeta function of $-v$. The reason that we look at $-v$ comes from the relation to the Novikov complex. In our general situation the Novikov complex of ω and v need not be defined, but we can use Theorem 1.1 for an approximation result, see Theorem 4.7 for details.

The definition of a noncommutative zeta function is algebraically more involved than that of a commutative zeta function (here commutative means that we consider closed orbits as defining homology classes instead of conjugacy classes of the fundamental group). Geoghegan and Nicas [6] were the first to attack this problem using Hochschild homology. Later Pajitnov [13] defined a noncommutative eta function for gradients of circle valued Morse functions which generalized the logarithm of the commutative zeta function. In [15] we introduced the zeta function which is also used here based on [6] and showed that it maps to the eta function of [13]. It was left open in [15] whether the zeta function can carry more information than the eta function. We show in Section 5 that they carry the same information.

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2. Preliminaries

Let M be a closed connected smooth manifold and $\xi \in H^1(M;\mathbb{R})$. By the universal coefficient theorem ξ is in 1-1 correspondence with homomorphisms $\xi : \pi_1(M) \to \mathbb{R}$. Furthermore every such homomorphism is represented by closed 1-forms ω on M. If ω is a closed 1-form, denote by $[\omega]$ the cohomology class.

Definition 2.1. Let M and ξ be as above.

(1) A smooth vector field v is called *gradient-like with respect to* ξ, if there exists a closed 1-form ω such that $[\omega] = \xi$, $\omega_x(v(x)) > 0$ if $\omega_x \neq 0$ and $v(x) = 0$ if $\omega_x = 0$ for all $x \in M$. In this case ω is called a *Lyapunov form* of v.

(2) The gradient-like vector field v is called *nice*, if there is a neighborhood U of $v^{-1}(0)$ in M with finitely many components so that $i^*\xi = 0 \in H^1(U;\mathbb{R})$ and the inclusion $v^{-1}(0) \subset U$ induces an isomorphism on π_0.

(3) A *homoclinic cycle* of a nice gradient-like vector field v is a sequence of nontrivial trajectories $\gamma_1, \ldots, \gamma_k$ of v such that there exist components $A_1, \ldots, A_k = A_0$ of $v^{-1}(0)$, so that for all $i \in \{1, \ldots, k\}$ we have $\gamma_i(t) \to A_i$ for $t \to \infty$ and $\gamma_i(t) \to A_{i-1}$ for $t \to -\infty$.

In the case where the homoclinic cycle only consists of one trajectory we also call it a *homoclinic orbit*.

Given a homoclinic cycle γ of a nice gradient-like vector field v of ξ notice that $\int_{\gamma_i} \omega$ is a positive real number and we define the length of γ by

$$l(\gamma) = \sum_{i=1}^{k} \int_{\gamma_i} \omega > 0.$$

This does not depend on the Lyapunov form ω. To see this let U be the neighborhood of $v^{-1}(0)$ as in the definition of a nice gradient-like vector field. For every γ_i choose $a_i < 0$ and $b_i > 0$ such that $\gamma_i(t) \in U$ for all $t < a_i$ and $t > b_i$. Now connect $\gamma_i(b_i)$ with $\gamma_{i+1}(a_{i+1})$ by a path in U. Using these paths and the trajectories from $\gamma_i(a_i)$ to $\gamma_i(b_i)$ we get a loop γ' and it is easy to see that $\xi([\gamma'])$ does not depend on the choices and equals $l(\gamma)$.

We denote by $\mathcal{GL}(\xi)$ the set of gradient-like vector fields v with respect to ξ and we define a subset

$$\mathcal{G}(\xi) = \{v \in \mathcal{GL}(\xi) \,|\, v \text{ is nice and has no homoclinic cycles}\}.$$

This is the set of vector fields for which we want to define zeta functions. We will often talk of gradient-like vector fields without specifying ξ if no confusion can arise.

Algebraic constructions.

Definition 2.2. Let G be a group and $\xi : G \to \mathbb{R}$ a homomorphism. We denote by $\mathbb{Z}^G$ the abelian group of all functions $G \to \mathbb{Z}$.

(1) For $\lambda \in \mathbb{Z}^G$ let $\operatorname{supp} \lambda = \{g \in G \,|\, \lambda(g) \neq 0\}$.
(2) The *Novikov ring* is defined to be $\widehat{\mathbb{Z}G}_\xi = \{\lambda \in \mathbb{Z}^G \,|\, \forall\, r \in \mathbb{R} \quad \operatorname{supp} \lambda \cap \xi^{-1}([r, \infty)) \text{ is finite}\}$. The multiplication is given as in the group ring $\mathbb{Z}G \subset \widehat{\mathbb{Z}G}_\xi$.

Instead of $\mathbb{Z}$ we can also use a different ring like $\mathbb{Q}$ or $\mathbb{R}$ in the above definition.

Definition 2.3. The *norm* of $\lambda \in \widehat{\mathbb{Z}G}_\xi$ is defined to be

$$\|\lambda\| = \|\lambda\|_\xi = \inf\{t \in (0, \infty) \,|\, \operatorname{supp} \lambda \subset \xi^{-1}((-\infty, \log t])\}.$$

Note that $\widehat{\mathbb{Z}G}_\xi$ is a completion of $\mathbb{Z}G$ with respect to the metric induced by this norm.

Let Γ be the set of conjugacy classes of G. Then ξ induces a well defined function $\Gamma \to \mathbb{R}$ which we also denote by ξ. In analogy to above we define $\widehat{\mathbb{Z}\Gamma}_\xi$, but since there is no well defined multiplication in Γ, this object is just an abelian group. Augmentation defines an epimorphism $\varepsilon : \widehat{\mathbb{Z}G}_\xi \to \widehat{\mathbb{Z}\Gamma}_\xi$ of abelian groups.

Let $W \subset \widehat{\mathbb{Z}G}_\xi$ be elements of the form $1 - a$ with $\|a\| < 1$. Then W is a subgroup of the group of units in $\widehat{\mathbb{Z}G}_\xi$ and $\|\cdot\|$ turns it into a topological group. Let $V = \ker(W \to K_1(\widehat{\mathbb{Z}G}_\xi))$. It is shown in [14, §3] that $L : W \to \widehat{\mathbb{Q}\Gamma}_\xi$ defined by $L(1-a) = -\sum_{k=1}^{\infty} \frac{\varepsilon(a^k)}{k}$ is a continuous homomorphism which vanishes on V. Denote the induced map by $\mathfrak{L} : W/V \to \widehat{\mathbb{Q}\Gamma}_\xi$.

Let $HH_*(\mathbb{Z}G)$ denote the Hochschild homology of the group ring. It is shown in Geoghegan and Nicas [5] that

$$HH_*(\mathbb{Z}G) \cong \bigoplus_{\gamma\in\Gamma} H_*(C(g_\gamma)),$$

where g_γ is a representative of $\gamma \in \Gamma$ and $C(g_\gamma)$ is the centralizer of g_γ. If $x \in \prod_{\gamma\in\Gamma} H_k(C(g_\gamma))$, then denote $x(\gamma) \in H_k(C(g_\gamma))$ the projection. Define $\operatorname{supp} x = \{\gamma \in \Gamma \,|\, x(\gamma) \neq 0\}$ and

$$\widehat{HH}_k(\mathbb{Z}G)_\xi = \{x \in \prod_{\gamma\in\Gamma} H_k(C(g_\gamma)) \,|\, \forall\, r \in \mathbb{R} \quad \operatorname{supp} x \cap \xi^{-1}([r,\infty)) \text{ is finite}\}.$$

Then $\widehat{HH}_*(\mathbb{Z}G)_\xi$ is a completion of $HH_*(\mathbb{Z}G)$ and in [15] natural homomorphisms $\theta : HH_*(\widehat{\mathbb{Z}G_\xi}) \to \widehat{HH}_*(\mathbb{Z}G)_\xi$ and $l : \widehat{HH}_1(\mathbb{Z}G)_\xi \to \widehat{\mathbb{R}\Gamma}_\xi$ are constructed such that $l \circ \theta \circ DT(x) = \mathfrak{L}(x)$ for $x \in W/V$, where DT is the Dennis trace homomorphism $K_1(\widehat{\mathbb{Z}G_\xi}) \to HH_1(\widehat{\mathbb{Z}G_\xi})$. Denote the homomorphism $\theta \circ DT$ by $\mathfrak{DT} : W/V \to \widehat{HH}_1(\mathbb{Z}G)_\xi$.

3. The zeta function of a gradient-like vector field

Let v be a smooth vector field on M and $\Phi : M \times \mathbb{R} \to M$ be the corresponding flow. Assume we have $x \in M$ and $p > 0$ such that $\Phi(x,p) = x$. If $v(x) = 0$ we get $\Phi(x,p) = x$ for all $p \in \mathbb{R}$. If $v(x) \neq 0$ we define the set $\pi = \{(\Phi(x,t),p) \in M \times (0,\infty) \,|\, t \in \mathbb{R}\}$ to be a *closed orbit* of v with *period* p. We will sometimes identify π with the restriction $\Phi|_{\{x\}\times[0,p]}$ and write $\pi : [0,p] \to M$. This way a closed orbit defines a conjugacy class $\{\pi\}$ of the fundamental group of M and also a homology class $[\pi] \in H_1(M)$. Let us denote $G = \pi_1(M)$ and Γ the conjugacy classes of G.

Now let $\xi : G \to \mathbb{R}$ be a homomorphism and π a closed orbit of $-v$, where v is a gradient-like vector field with respect to ξ. Then

$$\xi(\{\pi\}) = \int_\pi \omega = \int_0^p \omega(\pi'(t))\,dt = -\int_0^p \omega(v(\pi(t)))\,dt < 0.$$

For positive integers k define

$$\mathcal{O}_k(-v) = \{\pi : [0,p] \to M \,|\, \pi \text{ is a closed orbit of } -v \text{ and } p \geq k\}$$

and

$$C_k(-v) = \sup\{c \in \mathbb{R} \,|\, -\xi(\{\pi\}) \geq c \text{ for all } \pi \in \mathcal{O}_k(-v)\} \in [0,\infty].$$

Since $\mathcal{O}_k(-v) \supset \mathcal{O}_{k+1}(-v)$ we get $C_k(-v) \to C(-v) \in [0,\infty]$.

Lemma 3.1. *If $v \in \mathcal{G}(\xi)$ then $C(-v) = \infty$.*

Proof. The proof is analogous to [15, Lm.5.7]. Assume that $C(-v) < \infty$. Then there is a sequence $\pi_k \in \mathcal{O}_k(-v)$ with $-\xi(\{\pi_k\}) \in [0, C(-v)]$ for all k.

Let ω be a Lyapunov form of v. For every component X of $v^{-1}(0)$ choose open sets U_X, V_X with $X \subset U_X \subset \overline{U}_X \subset V_X$ and $\overline{V}_X \cap \overline{V}_Y = \emptyset$ for $X \neq Y$. Since v is nice we can assume that ω is exact on V_X. Whenever a flowline of $-v$ leaves V_X it takes a positive time $t_0 > 0$ to get back into U_Y for a component Y of $v^{-1}(0)$. If π is a closed orbit, let N_π be the number of times the orbit enters U_X and leaves V_X. Note that if a closed orbit enters U_X, it will leave V_X since ω is exact on V_X and so

$-v$ cannot have closed orbits inside V_X. Now N_{π_k} is bounded because $t_0 > 0$ and there is an $\varepsilon > 0$ such that $\omega_x(v(x)) \geq \varepsilon$ for $x \notin U_X$. So if N_{π_k} were not bounded, we would get $-\xi(\{\pi_k\}) \to \infty$. By passing to a subsequence we can assume that N_{π_k} is constant to N. Choose a point $x_{k,1} \notin V_X$ in the image of π_k, then follow the flowline until it enters U_X and leaves V_X for the first time. Choose a point $x_{k,2} \notin V_X$ in the image of π_k before the flowline enters U_X again. Continuing this way we get points $x_{k,j} \notin V_X$ on π_k for $j = 1, \ldots, N$ such that between $x_{k,j}$ and $x_{k,j+1}$ for $j = 1, \ldots, N-1$ and $x_{k,N}$ and $x_{k,1}$ the flowline enters U_X and leaves V_X exactly once. Denote by $t_{k,j}$ the time it takes from $x_{k,j}$ to $x_{k,j+1}$. By passing to a subsequence of the π_k we can assume that the $x_{k,j}$ converge to $x_j \in M$ and the $t_{k,j}$ converge to $t_j \in [0, \infty]$. Notice that $\sum_{j=1}^{N} t_{k,j} = p(\pi_k)$, the period of π_k. If $t_j < \infty$ the continuity of the flow implies the existence of a flowline between x_j and x_{j+1}. If $t_j = \infty$ there is a broken flowline from x_j to x_{j+1} through a component of zeros of v. At least one of the t_j has to be ∞ because $p(\pi_k) \to \infty$. Therefore there exists a homoclinic cycle contradicting $v \in \mathcal{G}(\xi)$. $\square$

Corollary 3.2. *Let $v \in \mathcal{G}(\xi)$. For every $c < 0$ there is a $k > 0$ such that $\pi \in \mathcal{O}_k(-v)$ implies $\xi(\{\pi\}) < c$.*

Let $F_k : M \times [0, k] \to M$ be the restriction of Φ. Geoghegan and Nicas [5] define the one-parameter trace

$$R(F_k) \in HH_1(\mathbb{Z}G) \cong \bigoplus_{\gamma \in \Gamma} C(g_\gamma)_{ab}.$$

We actually define $R(F_k)$ as in [15] which uses a different sign convention as [5]. The one-parameter trace has the property that $R(F_k)(\gamma) \neq 0$ implies the existence of a closed orbit π with $\{\pi\} = \gamma$, compare [15, §5].

Definition 3.3. Let $v \in \mathcal{G}(\xi)$. Then the *noncommutative zeta function* of $-v$ is defined as

$$\zeta(-v) = \lim_{k \to \infty} R(F_k) \in \widehat{HH}_1(\mathbb{Z}G)_\xi.$$

By Corollary 3.2 and [15, Lm.5.4] we get that $R(F_k)$ is a Cauchy sequence so it converges in $\widehat{HH}_1(\mathbb{Z}G)_\xi$.

Let us show that $\zeta(-v)$ contains some recognizable information. If $\gamma \in \Gamma$, let $C_\gamma \subset M \times (0, \infty)$ be the union of closed orbits π with $\{\pi\} = \gamma$. By Corollary 3.2 and Fuller [4, Th.3] the Fuller index $i(C_\gamma)$ is well defined.

Definition 3.4. Let $v \in \xi$. Then the *noncommutative eta function* $\eta(-v) \in \widehat{\mathbb{Q}\Gamma}_\xi$ is defined by $\eta(-v)(\gamma) = i(C_\gamma)$.

Using Geoghegan and Nicas [6, Th.2.7] it follows as in [15, §5] that

$$l(\zeta(-v)) = \eta(-v).$$

The noncommutative eta function first appeared in Pajitnov [13] while a noncommutative zeta function based on Hochschild homology classes already appeared in Geoghegan and Nicas [6]. Our zeta function draws a connection between these objects in that it detects both objects. Note that the eta function is detected by the zeta function and there is no obvious map $\widehat{\mathbb{Q}\Gamma}_\xi \to \widehat{HH}_1(\mathbb{Z}G)_\xi$ which sends the eta function to the zeta function. Nevertheless we show in Section 5 that the eta function carries the same information as the zeta function.

Let us discuss the commutative case which predates the noncommutative one. For $\delta \in H_1(M) = G_{ab}$ let C_δ be the union of closed orbits π with $[\pi] = \delta$. Then the *commutative eta function* $\bar{\eta}(-v) \in \widehat{\mathbb{Q}G_{ab}}_\xi$ is defined by $\bar{\eta}(-v)(\delta) = i(C_\delta)$. Notice that $\|\bar{\eta}(-v)\| < 1$. Thus $\bar{\zeta}(-v) = \exp \bar{\eta}(-v)$ is well defined, where exp is the usual power series, and we call $\bar{\zeta}(-v)$ the commutative zeta function. This formula already appeared in Fried [3] for nonsingular gradient-like vector fields. Clearly $\varepsilon(\eta(-v)) = \bar{\eta}(-v)$ for the augmentation $\varepsilon : \widehat{\mathbb{Q}\Gamma}_\xi \to \widehat{\mathbb{Q}G_{ab}}_\xi$, so $\exp \circ \varepsilon \circ l(\zeta(-v)) = \bar{\zeta}(-v)$.

4. Properties of the zeta function

We want to start by showing that $\zeta(-v)$ depends continuously on the vector field. This was shown in [15, §8] for gradients of a fixed Morse closed 1-form ω, but the methods carry over to the more general case. We equip $\mathcal{GL}(\xi)$ and $\mathcal{G}(\xi)$ with the C^0-topology. This turns out to be sufficient since by [1, §4,Th.3] trajectories of smooth vector fields depend continuously on the vector fields in the C^0-topology.

Definition 4.1. Let v be a gradient-like vector field with respect to ξ.

(1) An *exact cover* of v is an open set U containing $v^{-1}(0)$ such that there is a Lyapunov form ω for v which is exact on U.

(2) A *U-cycle* of an exact cover U is a finite sequence of trajectories $\gamma_j : [a_j, b_j] \to M$, $j = 1, \dots, k$ which start and end in U and such that $[\gamma_j(b_j)] = [\gamma_{j+1}(a_{j+1})]$ for $j = 1, \dots, k-1$ and $[\gamma_k(b_k)] = [\gamma_1(a_1)]$. Here $[x]$ denotes component of x in U. We write $\gamma = (\gamma_j)_{j=1}^k$.

Remark 4.2. (1) Every nice gradient-like vector field admits an exact cover U. Furthermore every homoclinic cycle gives rise to a U-cycle.

(2) If v admits an exact cover U, every gradient-like vector field w C^0-close enough to v admits the same exact cover U.

As with the homoclinic cycles we can define the length of a U-cycle γ. To do this we need to connect $\gamma_j(b_j)$ with $\gamma_{j+1}(a_{j+1})$ within U to get a cycle. Since ω is exact on U the image of this cycle under ξ is well defined and we denote it by $l(\gamma)$. Unlike in the case of a homoclinic cycle it is possible that $l(\gamma) \leq 0$, for example this happens for a small trajectory that never leaves U. The existence of U-cycles γ with $l(\gamma) < 0$ can be avoided for nice gradient-like vector fields by choosing the cover U small enough. A U-cycle γ is called *nondegenerate*, if it does not contain any sub-U-cycles of length ≤ 0.

Definition 4.3. Let $U \subset M$ be open.

(1) If v is a gradient-like vector field such that U is an exact cover of v, then $b_\xi^U(-v) = \sup\{-l(\gamma) \in (-\infty, 0) \,|\, \gamma \text{ is a nondegenerate } U\text{-cycle}\}$.

(2) Let $r \in (-\infty, 0)$ and $(v_t)_{t \in [0,1]}$ be a smoothly varying one parameter family of gradient-like vector fields such that U is an exact cover for v_t for every $t \in [0.1]$. We say (v_t) is (R, U)-controlled, if $b_\xi^U(-v_t) < R$ for all $t \in [0, 1]$.

Note that $b_\xi^U(v) = -\infty$ if and only if there are no nondegenerate U-cycles.

Proposition 4.4. *Let $R \in (-\infty, 0)$, $U \subset M$ open and $(v_t)_{t \in [0,1]}$ be an (R, U)-controlled one parameter family of gradient-like vector fields such that $v_0, v_1 \in \mathcal{G}(\xi)$. Then $\zeta(-v_0)(\gamma) = \zeta(-v_1)(\gamma) \in C(g_\gamma)_{ab}$ for every $\gamma \in \Gamma$ with $\xi(\gamma)R$.*

Proof. The proof is a generalization of the proof of [15, Prop.8.2] just as Lemma 3.1 generalized [15, Lm.5.7]. □

Lemma 4.5. *Let $v \in \mathcal{G}(\xi)$ and $R \in (-\infty, 0)$. Then there exists an exact cover U of v and a neighborhood $\mathcal{V}$ of v in $\mathcal{GL}(\xi)$ such that $b_\xi^U(-w) < R$ for all $w \in \mathcal{V}$.*

Proof. Since v is a nice gradient-like vector field, there exists an exact cover U of v with finitely many components. By possibly shrinking U we can assume that $b_\xi^U(-v) < R$, for if not an argument as in Lemma 3.1 detects a homoclinic cycle.

As mentioned before every gradient-like vector field near v has U as an exact cover. Let $U_1, \ldots, U_k$ be the components of U. By a U-path from U_i to U_j we mean finitely many trajectories $\gamma_l : [a_l, b_l] \to M$ of $-v$ for $l = 1, \ldots, m$ such that $\gamma_1(a_1) \in U_i$, $\gamma_{m+1}(b_{m+1}) \in U_j$ and $\gamma_l(b_l)$ and $\gamma_{l+1}(a_{l+1})$ are in the same component of U for all l. We write γ for such a U-path. We can connect each $\gamma_l(b_l)$ and $\gamma_{l+1}(a_{l+1})$ within U and then get $\int_\gamma \omega$ to be a well defined number. For $i, j \in \{1, \ldots, k\}$ with $i \neq j$ let $\mathcal{P}_{ij}(v, U) = \{\gamma = (\gamma_1, \ldots, \gamma_m) \,|\, \gamma \text{ is a } U\text{-path of } -v \text{ with } \gamma(a_1) \in U_i, \gamma(b_m) \in U_j\}$. Also define $\mathcal{P}_{ii}(v, U)$ analogously with the extra assumption that γ defines a nondegenerate U-cycle. Now let $m_{ij}(v, U) = \sup\{\int_\gamma \omega \in (-\infty, 0) \,|\, \gamma \in \mathcal{P}_{ij}(v, U)\}$ where ω is a Lyapunov form of v. Note that ω can be used as a Lyapunov form outside of U for vector fields close enough to v. We have $m_{ij}(v, U) < 0$ and $m_{ii}(v, U) < R$.

Choose a metric d on M and let $C = v^{-1}(\{0\})$. Let $\varepsilon > 0$ be so small that $d(x, y) > 3\varepsilon$ holds for all $x \in C$ and $y \in M - U$. Now let $U_\varepsilon = \{x \in U \,|\, \text{there is } y \in C \text{ such that } d(x, y) < \varepsilon\}$. Then $C \subset U_\varepsilon \subset U_{2\varepsilon} \subset U$. We can assume that the ε is chosen so small that U_ε and $U_{2\varepsilon}$ both have k components given by $U_{\varepsilon,j} = U_\varepsilon \cap U_j$ and $U_{2\varepsilon,j} = U_{2\varepsilon} \cap U_j$ for $j = 1, \ldots, k$.

For $t \geq 0$ and $x \in M$ let $\gamma_{x,t} : [0, t] \to M$ be the restriction of the trajectory of $-v$ starting at x.

We claim there is a $T > 0$ such that for every $x \in M$ we either have $\int_{\gamma_{x,T}} \omega \leq R - 1$ or there is a t with $0 \leq t \leq T$ such that $\gamma_{x,T}(t) \in U_\varepsilon$. To see this note that there is a $\delta > 0$ such that $\omega_x(v(x)) \geq \delta$ for all $x \in M - U_\varepsilon$. Now it is easy to see that $T = \frac{-R+1}{\delta}$ works.

Since trajectories of smooth vector fields depend continuously on the vector field in the C^0-topology and $M \times [0, T]$ is compact, we can find a neighborhood $\mathcal{V}$ of v such that for every $w \in \mathcal{V}$ we have $\omega_x(w(x)) > 0$ for $x \in M - U_\varepsilon$ and $d(\gamma_{x,T}(t), \gamma'_{x,T}(t)) < \varepsilon$ for $t \in [0, T]$. Here $\gamma'_{x,T}$ is the trajectory of $-w$. It is easy to see that we can find a Lyapunov form for every $w \in \mathcal{V}$ which agrees with ω outside U_ε and so that $U_{2\varepsilon}$ is an exact cover of w.

We claim that $m_{ij}(w, U_{2\varepsilon}) \leq \max\{R, m_{ij}(v, U)\}$.

Let $\gamma \in \mathcal{P}_{ij}(w, U_{2\varepsilon})$ with $\int_\gamma \omega R$. We want to show there is a $\gamma' \in \mathcal{P}_{ij}(v, U)$ with $\int_{\gamma'} \omega \geq \int_\gamma \omega$.

Let $\gamma = (\gamma_1, \ldots, \gamma_l)$ with $\gamma_m : [a_m, b_m] \to M$ having the property that $\gamma_m([a_m, b_m]) \cap U_{2\varepsilon} \subset U_{2\varepsilon,s} \cup U_{2\varepsilon,t}$ for $s, t \in \{1, \ldots, k\}$ so that $\gamma_m(a_m) \in U_{2\varepsilon,s}$ and $\gamma_m(b_m) \in U_{2\varepsilon,t}$. Notice that if γ does not have this property we can find a $U_{2\varepsilon}$-path $\bar{\gamma}$ with this property and with $\int_\gamma \omega \geq \int_{\bar{\gamma}} \omega$ by splitting γ_m into two or more trajectories.

Look at $\gamma_m : [a_m, b_m] \to M$. Let $t_0 = \sup\{t \in [a_m, b_m] \,|\, \gamma_m(t) \in \bar{U}_{2\varepsilon,s}\}$ and $t_1 = \inf\{t \in [a_m, b_m] \,|\, \gamma_m(t) \in \bar{U}_{2\varepsilon,t}\}$. We can assume that $t_0 = 0$. If $t_1 > T$, we get $d(\gamma_m(t), \gamma_v(t)) < \varepsilon$ for $0 \leq t \leq T$, where γ_v is the trajectory of $-v$ having the property that $\gamma_v(0) = \gamma_m(0)$. Since γ_m does not enter $U_{2\varepsilon}$ between 0 and T, we get that γ_v does not enter U_ε. By the choice of T we get $\int_{\gamma_v|[0,T]} \omega < R - 1$ and then $\int_\gamma \omega < R$ contrary to the choice of γ.

Therefore we can assume that $t_1 \leq T$. But now $d(\gamma_m(t), \gamma_v(t)) < \varepsilon$ for $0 \leq t \leq t_1$ and in particular $\gamma_v(t_1) \in U$. Therefore we get a U-path γ' which stays close to the $U_{2\varepsilon}$-path γ. It follows that $\int_{\gamma'} \omega \geq \int_\gamma \omega$ and the claim follows.

But from $m_{ij}(U_{2\varepsilon}, w) < \max\{R, m_{ij}(v, U)\}$ it follows easily that $b_\xi^{U_{2\varepsilon}}(-w) < R$ and the lemma is proven. □

Theorem 4.6. *Let $\xi : G \to \mathbb{R}$ be a homomorphism where G is the fundamental group of the closed connected smooth manifold M. Then $\zeta : \mathcal{G}(\xi) \to \widehat{HH}_1(\mathbb{Z}G)_\xi$ sending v to $\zeta(-v)$ is continuous.*

Proof. Let $v \in \mathcal{G}(\xi)$ and $R \in (-\infty, 0)$. By Proposition 4.4 it is enough to find an open set U such that every gradient-like vector field w near v can be connected to v by an (R, U)-controlled one parameter family. We get U from Lemma 4.5.

It remains to connect v and w by a one parameter family of gradient-like vector fields within $\mathcal{V}$. Recall the Lyapunov form ω of v. Let v' be a vector field which is dual to ω with respect to a Riemannian metric g in an exact cover $U' \subset U$ and agrees with v outside of U. Then $tv + (1-t)v'$ has Lyapunov form ω for every $t \in [0, 1]$. Do the same with w to get w' dual to an ω' on U' with respect to the same Riemannian metric. For every $t \in [0, 1]$ $t\omega + (1-t)\omega'$ is a closed 1-form cohomologous to ω. On U' the gradient of this form can be used to define a path between v' and a vector field v'' which agrees with w' on U' and with v' outside of U. Now use $tv'' + (1-t)w'$. These paths can be combined to define a one parameter family between v and w which is (R, U)-connected for v and w close enough. By Proposition 4.4 the result follows. □

Relations to the Novikov complex. Let ω be a closed 1-form which has only nondegenerate zeros. We call such forms *Morse forms.* If v is a vector field that is dual to ω with respect to a Riemannian metric, then the stable and unstable manifolds $W^s(p, v)$ and $W^u(p, v)$ are injectively immersed submanifolds of M for every $p \in M$ with $\omega_p = 0$. The stable manifolds are of dimension $\operatorname{ind} p$ and the unstable ones of dimension $\dim M - \operatorname{ind} p$. If all of these stable and unstable manifolds intersect transversely, we can define the Novikov complex $C_*(\tilde{M}, \omega, v)$, see for example [10, 14]. This is a free $\widehat{\mathbb{Z}G}_\xi$ complex generated by the zeros of ω and graded by the index which is chain homotopy equivalent to $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M})$, the latter being the simplicial chain complex of the universal cover of M coming from a smooth triangulation of M. Relations between the Novikov complex and zeta functions have appeared already in Hutchings and Lee [8], Hutchings [7], Pajitnov [11, 13] and the author [14, 15]. In [14, 15] a natural chain homotopy equivalence $\varphi_v : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \to C_*(\tilde{M}, \omega, v)$ is constructed such that its torsion $\tau(\varphi_v) \in W/V$ and

$$\mathfrak{D}\mathfrak{T}(\tau(\varphi_v)) = \zeta(-v). \tag{1}$$

In the situation where $v \in \mathcal{G}(\xi)$, we do not have a Novikov complex in general, but we can combine (1) and Theorem 4.6 to an approximation result.

Let $\mathcal{GT}(\xi) \subset \mathcal{GL}(\xi)$ be the set of vector fields v which are dual to a Morse form and such that all stable and unstable manifolds intersect transversely. Then we get in fact $\mathcal{GT}(\xi) \subset \mathcal{G}(\xi)$, since trajectories of $-v$ between critical points have to decrease the index by the transversality condition. Combining a version of the Kupka-Smale theorem, see Pajitnov [12, Lm.5.1] for a convenient formulation, with density results of Morse forms, see Milnor [9, §2], we get that $\mathcal{GT}(\xi)$ is dense in $\mathcal{GL}(\xi)$. So given $v \in \mathcal{G}(\xi)$ we can find a sequence $v_n \in \mathcal{GT}(\xi)$ with $v_n \to v$ and hence

$$\lim_{n\to\infty} \mathfrak{DT}(\tau(\varphi_v)) = \zeta(-v).$$

Let us write down a similar approximation result.

Theorem 4.7. *Let $v \in \mathcal{G}(\xi)$, $R \in (-\infty, 0)$ and U an exact cover with $b^U_\xi(-v) < R$. Then there exists a $w \in \mathcal{GT}(\xi)$ with $w|_{M-U} = v|_{M-U}$ and $\mathfrak{DT}(\tau(\varphi_w))(\gamma) = \zeta(-v)(\gamma)$ for $\gamma \in \Gamma$ with $\xi(\gamma) \geq R$.*

Proof. Let ω be a Lyapunov form of v and V an exact cover of v with $\overline{V} \subset U$. The techniques of Milnor [9, §2] give a Morse form ω' that agrees with ω on $M - V$. On $M - V$ we have $\omega(v) \geq \varepsilon$ for some $\varepsilon > 0$. Therefore we can find local coordinates such that ω looks locally like the differential of $f(x_1, \ldots, x_n) = x_1$. Now it is easy to find a Riemannian metric for these coordinates such that v is dual to ω on these coordinates. By glueing them together we get a Riemannian metric on $M - V$ such that v is dual to ω. Extend the Riemannian metric to M using U and let v' be dual to ω'. We have $v' = v$ on $M - U$. This v' does not have to satisfy the transversality condition, but we can find $w \in \mathcal{GT}(\xi)$ with $w|_{M-U} = v'|_{M-U} = v|_{M-U}$. We now get $\zeta(-w)(\gamma) = \zeta(-v)(\gamma)$ for $\gamma \in \Gamma$ with $\xi(\gamma) \geq R$ as before and the result follows by (1). □

Note that given $v \in \mathcal{G}(\xi)$ and $R < 0$ we can always find an exact cover U such that $b^U_\xi(-v) < R$ by Lemma 4.5.

5. Comparing the zeta and eta functions

In the commutative case the zeta and eta functions carry the same information since we have $\bar{\zeta}(-v) = \exp \bar{\eta}(-v)$ and $\bar{\eta}(-v) = \log \bar{\zeta}(-v)$. In the noncommutative case we have $l(\zeta(-v)) = \eta(-v)$ but it is not clear how to define a homomorphism $e : \widehat{\mathbb{Q}\Gamma}_\xi \to \widehat{HH}_1(\mathbb{Z}G)_\xi$ with $e(\eta(-v)) = \zeta(-v)$. In [15, §11] a rational zeta function $\zeta_\mathbb{Q}(-v) \in \widehat{HH}_1(\mathbb{Q}G)_\xi$ and a homomorphism $e : \widehat{\mathbb{Q}\Gamma}_\xi \to \widehat{HH}_1(\mathbb{Q}G)_\xi$ are defined such that $e(\eta(-v)) = \zeta_\mathbb{Q}(-v)$. Here $\widehat{HH}_1(\mathbb{Q}G)_\xi$ is a completion of $\mathbb{Q} \otimes HH_1(\mathbb{Z}G)$ and there is a natural map $\hat{\imath} : \widehat{HH}_1(\mathbb{Z}G)_\xi \to \widehat{HH}_1(\mathbb{Q}G)_\xi$ with $\hat{\imath}(\zeta(-v)) = \zeta_\mathbb{Q}(-v)$. The problem whether $\zeta(-v)$ carries more information than $\zeta_\mathbb{Q}(-v)$ or $\eta(-v)$ was left open in [15]. Now we will show that $\zeta(-v)$ does not carry more information. The arguments are in fact quite similar to Pajitnov [13, §3].

Recall the groups W and V from Section 2. The homomorphism $L : W \to \widehat{\mathbb{Q}\Gamma}_\xi$ defined by $L(1-a) = -\sum \frac{\varepsilon(a^n)}{n}$ vanishes on V. Since L is continuous, it also vanishes on $\overline{V}$, the closure of V in the topological group W. Denote the resulting map by $\overline{\mathfrak{L}} : W/\overline{V} \to \widehat{\mathbb{Q}\Gamma}_\xi$. Let $\tau : W \to W/V$ and $\overline{\tau} : W \to W/\overline{V}$ denote the quotient maps.

Theorem 5.1. $\overline{\mathfrak{L}} : W/\overline{V} \to \widehat{\mathbb{Q}\Gamma}_\xi$ *is injective.*

The main step to prove the theorem will be the next lemma.

Lemma 5.2. *Let $1-a \in W$ and $R \in (0,1)$ a real number such that $\|L(1-a)\| < R$. Then there is a $1-a' \in W$ with $\tau(1-a') = \tau(1-a)$ and $\|a'\| \leq R$.*

Proof. Since $a \in \widehat{\mathbb{Z}G}_\xi$ there is a finite sequence of numbers $R_1, \ldots, R_k$ with $R < R_1 < \ldots < R_k < 1$ so that we can write $a = a_0 + a_1 + \ldots + a_k$ with $\|a_0\| \leq R$ and $\operatorname{supp} a_i \subset \xi^{-1}(\{\log R_i\})$ for $i = 1, \ldots, k$. We can assume that for every i, j with $R_i R_j > R$ there is an $l \in \{1, \ldots, k\}$ with $R_l = R_i R_j$. Of course we then have to allow that $a_i = 0$ for some i.
We have $L(1-a) = -\varepsilon(a_k) + x$ with $\|x\| \leq R_{k-1}$. Note that $R_k^2 \leq R_{k-1}$ by assumption. Since $\|L(1-a)\| < R$ we get $\varepsilon(a_k) = 0$. Therefore there exist $n_{ij} \in \mathbb{Z}$, $g_j, h_{ij} \in G$ with $a_k = \sum_{i,j} n_{ij} h_{ij}^{-1} g_j h_{ij}$ with $\sum_i n_{ij} = 0$ and $\xi(g_j) = \log R_k$.
A straightforward calculation shows that

$$\prod_{i,j}(1 - h_{ij}^{-1} g_j h_{ij})^{n_{ij}} = 1 - a_k - y$$

with $\operatorname{supp} y \subset \xi^{-1}(\{R_k^l \,|\, l \geq 2\}) \subset \xi^{-1}((-\infty, R] \cup \{R_1, \ldots, R_{k-1}\})$. Since W/V is a subgroup of $K_1(\widehat{\mathbb{Z}G}_\xi)$ we get

$$\tau\left(\prod_{i,j}(1 - h_{ij}^{-1} g_j h_{ij})^{n_{ij}}\right) = \tau\left(\prod_{i,j}(1 - g_j)^{n_{ij}}\right) = \tau\left(\prod_{j}(1 - g_j)^{\sum_i n_{ij}}\right) = 0.$$

Now

$$1 - a = (1-a)(1 - a_k - y)^{-1}(1 - a_k - y) = (1 - z)(1 - a_k - y)$$

with $\operatorname{supp} z \subset \xi^{-1}((-\infty, R] \cup \{R_1, \ldots, R_{k-1}\})$. Note that the support condition follows from the fact that either $R_i R_j < R$ or $R_i R_j = R_l$ for some $l \in \{1, \ldots, k-1\}$. In particular we get $\tau(1-a) = \tau(1-z)$ and $\|z\| \leq R_{k-1}$. Induction on k gives the result. □

Proof of Theorem 5.1. As mentioned above $\overline{V} \subset \ker L$. It remains to show that $\ker L \subset \overline{V}$. So let $1 - a \in W$ satisfy $L(1-a) = 0$. By Lemma 5.2 there is a sequence $1 - a_n \in W$ with $\tau(1-a_n) = \tau(1-a)$ and $\|a_n\| < \frac{1}{n}$. Now $W/\overline{V}$ is a Hausdorff space, so

$$0 = \overline{\tau}(1) = \lim_{n\to\infty} \overline{\tau}(1 - a_n) = \overline{\tau}(1-a),$$

so $1 - a \in \overline{V}$. □

Corollary 5.3. *The natural map $l : \widehat{HH}_1(\mathbb{Z}G)_\xi \to \widehat{\mathbb{R}\Gamma}_\xi$ restricts to an injective map $l| : \operatorname{im}\mathfrak{DT} \to \widehat{\mathbb{Q}\Gamma}_\xi$.*

Proof. Since $\mathfrak{DT} \circ \tau : W \to \widehat{HH}_1(\mathbb{Z}G)_\xi$ is continuous and vanishes on V, it vanishes on $\overline{V}$. Let $\overline{\mathfrak{DT}}$ be the induced map $W/\overline{V} \to \widehat{HH}_1(\mathbb{Z}G)_\xi$. Then $l \circ \overline{\mathfrak{DT}} = \overline{\mathfrak{L}}$ and $\operatorname{im}\mathfrak{DT} = \operatorname{im}\overline{\mathfrak{DT}}$. □

In particular $\overline{\mathfrak{DT}} : W/\overline{V} \to \widehat{HH}_1(\mathbb{Z}G)_\xi$ is also injective.

Corollary 5.4. $\operatorname{im}\mathfrak{DT}$ *is closed in* $\widehat{HH}_1(\mathbb{Z}G)_\xi$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{im}\mathfrak{D}\mathfrak{T}$ which converges to $x \in \widehat{HH}_1(\mathbb{Z}G)_\xi$. Then there are $1-a_n \in W$ with $x_n = \mathfrak{D}\mathfrak{T}(\tau(1-a_n))$. Define a sequence $1-b_n$ by $1-b_1 = 1-a_1$, $b_i = (1-x_i)(1-b_{i-1})$ for $i \geq 2$ where $\tau(1-x_i) = \tau((1-a_i)(1-a_{i-1})^{-1})$ and $\|x_i\| < \|L((1-a_i)(1-a_{i-1})^{-1})\| + \frac{1}{i}$. Such x_i exist by Lemma 5.2. By induction we see that $\tau(1-b_i) = \tau(1-a_i)$. Let k be a positive integer, then

$$\begin{aligned} \|1-(1-b_i)\cdot(1-b_{i+k})^{-1}\| &= \|1-\prod_{j=1}^{k}(1-x_{i+j})^{-1}\| \\ &\leq \max\{\|L((1-a_{i+j})(1-a_{i+j-1})^{-1})\| + \frac{1}{i}\}. \end{aligned}$$

Since $(L(1-a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, it follows that $(1-b_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence. But W is complete, so we get a $1-b \in W$ with $\mathfrak{D}\mathfrak{T}(\tau(1-b)) = x$. □

It follows that the noncommutative eta function carries the same information as the noncommutative zeta function. By combining Theorem 4.7 with Corollary 5.4 we also see that $\zeta(-v) \in \operatorname{im}\mathfrak{D}\mathfrak{T}$ for $v \in \mathcal{G}(\xi)$.

6. Concluding Remarks

If the homomorphism $\xi : G \to \mathbb{R}$ satisfies certain nice group theoretic conditions, for example $\ker\xi$ being finitely presented, we get that $\zeta(\mathcal{G}(\xi))$ is dense in $\operatorname{im}\mathfrak{D}\mathfrak{T} \subset \widehat{HH}_1(\mathbb{Z}G)_\xi$. This follows from [16, Th.1.3].

It might be possible to define zeta functions for gradient-like vector fields which are not nice, but the condition that there are no homoclinic cycles cannot be dropped, at least if the zeta function should depend continuously on the vector fields. Using [14, Ex.5.3] and [14, Rm.5.4] it is easy to construct a gradient-like vector field v on the surface of genus 2 with a homoclinic cycle such that ζ does not extend continuously to v, even if we use the C^∞-topology.

Finally let us discuss the existence of gradient-like vector fields which have no homoclinic cycles. If we just look at gradient-like vector fields which are gradient to a Morse form ω with respect to some Riemannian metric, the condition that there are no homoclinic cycles is in fact a generic condition. The reason is that the unstable and stable manifolds are actually injectively immersed submanifolds and by the techniques of the Kupka-Smale theorem these manifolds intersect transversely for a generic set of vector fields. But if the vector field is allowed to have nondegenerate zeros these techniques no longer apply. Furthermore Farber [2] has defined a Lusternik-Schnirelman theory for finite CW-complexes X and $\xi \in H^1(X;\mathbb{R})$ which assigns the pair (X,ξ) a nonnegative integer $\operatorname{cat}(X,\xi)$ which is a homotopy invariant. He shows that every gradient-like vector field with respect to ξ on the closed connected smooth manifold M which has less than $\operatorname{cat}(M,\xi)$ zeros has to have a homoclinic cycle. He also shows that for $\xi \in H^1(X;\mathbb{Z})$ there is a closed 1-form ω with at most one zero. So if $\operatorname{cat}(M,\xi) \geq 2$, every gradient-like vector field with such a Lyapunov form has a homoclinic cycle.

This raises the question what the minimal number of zeros of a closed 1-form ω with $[\omega] = \xi$ is such that it admits gradient-like vector fields without homoclinic cycles. In [17] we investigate this question and show how to construct many examples of gradient-like vector fields with degenerate zeros and no homoclinic cycles. The basic idea is to take a Morse form ω and push different zeros into one degenerate zero using the techniques of Takens [18]. Under certain conditions it is possible to do this without introducing homoclinic cycles.

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