GEOMETRIC CHAIN HOMOTOPY EQUIVALENCES BETWEEN NOVIKOV COMPLEXES

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Abstract. We give a detailed account of the Novikov complex corresponding to a closed 1-form ω on a closed connected smooth manifold M. Furthermore we deduce the simple chain homotopy type of this complex using various geometrically defined chain homotopy equivalences and show how they are related to another.

1. Introduction

The purpose of this article is to give a detailed exposition of the Novikov complex as defined in Novikov [11], and its chain homotopy type. Given a closed 1-form ω on a closed connected smooth manifold M with only nondegenerate critical points, this complex is freely generated by the critical points over an appropriate ring. The grading of the complex is given by the indices of the critical points. To define the boundary operator a vector field v gradient to ω is required. The boundary is then given by counting the trajectories between critical points.

In the case where ω is exact this complex has already been described by Milnor [9, §7] and the ring can be chosen to be $\mathbb{Z}$ or a group ring $\mathbb{Z}G$, if a regular covering $\rho : \tilde{M} \to M$ with covering transformation group G is considered. But in the nonexact case the group ring $\mathbb{Z}G$ no longer works and we have to use a completion $\widehat{\mathbb{Z}G}_\xi$ of $\mathbb{Z}G$. Now G is the covering transformation group of a regular covering $\rho : \tilde{M} \to M$ such that ω pulls back to an exact form. The completion also depends on a homomorphism $\xi : G \to \mathbb{R}$ induced by ω.

As it turns out, the Novikov complex $C_*(\tilde{M}, \omega, v)$ is chain homotopy equivalent to $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*(\tilde{M})$, where $C_*(\tilde{M})$ is the singular chain complex of $\tilde{M}$. Even though a first version was already announced in [11], detailed proofs did not appear until much later, see Latour [8] or Pajitnov [12]. Since then easier proofs have appeared based on geometrically defined chain homotopy equivalences. These equivalences appear in various places in the literature, but are not very well connected to each other. Given a smooth triangulation Δ of M, there is a chain homotopy equivalence $\varphi_v : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C^\Delta_*(\tilde{M}) \to C_*(\tilde{M}, \omega, v)$ based on intersection numbers between simplices and unstable manifolds of critical points. Versions of this equivalence appeared in Hutchings and Lee [7] and Schwarz [24], and the torsion properties have been discussed in [22].

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Given another closed 1-form ω' cohomologous to ω with a gradient w there are different ways to describe chain homotopy equivalences $\psi_{v,w} : C_*(\tilde{M}, \omega, v) \to C_*(\tilde{M}, \omega', w)$, see, for example, Latour [8, §2.21] or Poźniak [19, §2.6]. We show that up to chain homotopy these definitions agree and that $\psi_{v,w}$ commutes with φ_v and φ_w.

The techniques in this paper are based on Milnor [9]. In Section 2 we recall some of the results of [9] and show how to obtain φ_v in the case of an ordinary Morse function. In Section 3 we look at the special case of a closed 1-form coming from a circle valued Morse function $f : M \to S^1$. It turns out that the results in the circle valued case can be reduced to the exact case by using inverse limit arguments. The general case of an arbitrary closed 1-form is then reduced to the circle valued case in Section 4 using approximation arguments. Section 5 shows how the principle of continuation as described in Poźniak [19, §2.6] fits into our description. Then we discuss the simple chain homotopy type of the Novikov complex. For the basic notion of torsion and simple chain homotopy type we refer the reader to Cohen [1]. It turns out that the torsion of all discussed equivalences are represented by so called trivial units. It has been known that these torsions carry information on the closed orbit structure of the gradients in form of zeta functions. We recall these results in Section 7. Finally we give an example of a gradient where the noncommutative zeta function contains more information than the commutative zeta function.

A lot of the results in this paper were already collected in [23, App.A], but in a very sketchy form. Here we give complete proofs which are new in certain cases.

Describing the Novikov complex as an inverse limit was already done in Pajitnov [12]; Latour [8] and Poźniak [19] use quite a different approach. A more algebraic method to obtain the Novikov complex is used in Farber [2], Farber and Ranicki [3] and Ranicki [20], but the boundary operator there is no longer described by trajectories between critical points in general.

If G is a group, we denote by $\mathbb{Z}^G$ the abelian group of all functions $\lambda : G \to \mathbb{Z}$. The group ring $\mathbb{Z}G \subset \mathbb{Z}^G$ consists of all $\lambda \in \mathbb{Z}^G$ such that $\lambda(g) = 0$ for all but finitely many $g \in G$. For $g \in G$ we define $\lambda_g \in \mathbb{Z}G$ by $\lambda_g(h) = 0$ for $h \neq g$ and $\lambda_g(g) = 1$. By abuse of notation we write $g = \lambda_g \in \mathbb{Z}G$.

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2. The Morse-Smale complex

Let $(W; M_0, M_1)$ be a compact cobordism between the closed manifolds M_0 and M_1. A Morse function f on W is a smooth function $f : W \to [0,1]$ such that $f^{-1}(\{0\}) = M_0$, $f^{-1}(\{1\}) = M_1$ and the critical points of f are in the interior and nondegenerate. It follows that f has only finitely many critical points and every such point $p \in W$ has an index denoted by $\operatorname{ind} p$. The set of critical points of index i will be denoted by $\operatorname{crit}_i(f)$. The existence of Morse functions is shown in Milnor [9, §3].

Let v be an f-gradient, i.e. a vector field dual to the exact 1-form df with respect to some Riemannian metric. Let Φ be the flow of v. The stable and unstable manifolds at a critical point are defined as

$$\begin{aligned} W^s(p,v) &= \{x \in W \mid \Phi(x,t) \to p \text{ for } t \to \infty\} \\ W^u(p,v) &= \{x \in W \mid \Phi(x,t) \to p \text{ for } t \to -\infty\} \end{aligned}$$

Then $W^s(p,v)$ is an immersed submanifold of dimension $\operatorname{ind} p$ and $W^u(p,v)$ one of dimension $n - \operatorname{ind} p$. If $W^s(p,v)$ and $W^u(q,v)$ intersect transversely for all critical points with $\operatorname{ind} p < \operatorname{ind} q$, we can find a filtration $M_0 = W_{-1} \subset W_0 \subset \ldots \subset W_n = W$ of W by compact cobordisms such that $W_i - \operatorname{int} W_{i-1}$ is a compact cobordism that contains exactly the critical points of index i in its interior. Furthermore if $x \in W_i$, so is $\Phi(x,t)$ for all $t < 0$ such that $\Phi(x,t)$ is defined.

Define $C_i(W, M_0, f, v) = H_i(W_i, W_{i-1})$ and $\partial : C_i(W, M_0, f, v) \to C_{i-1}(W, M_0, f, v)$ to be the connecting homomorphism of the triple (W_i, W_{i-1}, W_{i-2}). Then $C_i(W, M_0, f, v)$ is a free abelian group of rank $|\mathrm{crit}_i(f)|$ and the stable manifolds of the critical points represent the generators. Furthermore we have $H_*(C_*(W, M_0, f, v)) = H_*(W, M_0)$.

If $p : \tilde{W} \to W$ is a regular covering space with covering transformation group G, define $\tilde{A} = p^{-1}(A)$ for subsets $A \subset W$. Then let $C_i(\tilde{W}, \tilde{M}_0, f, v) = H_i(\tilde{W}_i, \tilde{W}_{i-1})$ and let the boundary be defined using the triple $(\tilde{W}_i, \tilde{W}_{i-1}, \tilde{W}_{i-2})$. Then $C_*(\tilde{W}, \tilde{M}_0, f, v)$ is a finitely generated free chain complex over $\mathbb{Z}G$ generated by the critical points of f and its homology is $H_*(\tilde{W}, \tilde{M}_0)$.

Notice that the chain groups are independent of the filtration. To express the boundary homomorphism in terms of the vector field alone, assume now also that $W^s(p,v)$ and $W^u(q,v)$ intersect transversely for all critical points of f with $\operatorname{ind} p \leq \operatorname{ind} q + 1$. For $\operatorname{ind} p = \operatorname{ind} q + 1 = i + 1$ we get that $W^s(p,v) \cap W^u(q,v) \cap \partial W_i$ is a compact 0-dimensional manifold. Therefore we have finitely many trajectories between p and q.

Definition 2.1. An f-gradient v is called *transverse,* if $W^s(p,v)$ and $W^u(q,v)$ intersect transversely for all critical points of f with $\operatorname{ind} p \leq \operatorname{ind} q + 1$.

Choose an orientation for every stable manifold $W^s(p,v)$ and orient the normal bundle of $W^u(p,v)$ such that the orientation of the normal bundle of $W^u(p,v)$ at p projects to the orientation of $W^s(p,v)$ at p. An orientation of the normal bundle will also be called a coorientation. Now let γ be a trajectory of $-v$ between p and q. For $t \in \mathbb{R}$ we have $df_{\gamma(t)}(\gamma'(t)) < 0$. Let $X_1, \ldots, X_i \in T_{\gamma(t)}M$ represent the coorientation of $W^u(q,v)$. If the projection of $\gamma'(t), X_1, \ldots, X_i$ into $T_{\gamma(t)}W^s(p,v)$ represents the orientation of $W^s(p,v)$, set $\varepsilon(\gamma) = 1$, otherwise set $\varepsilon(\gamma) = -1$. Note that the projection does represent a basis of $T_{\gamma(t)}W^s(p,v)$ by the transversality assumption.

Now lift the orientations to $\tilde{W}$ and choose for every critical point p of f exactly one lift $\tilde{p}$ in $\tilde{W}$. For critical points p, q with $\operatorname{ind} p = \operatorname{ind} q + 1$ define $[p : q] \in \mathbb{Z}G$ by

$$[p:q](g) = \sum \varepsilon(\gamma)$$

where the sum is taken over the finite set of trajectories between $\tilde{p}$ and $g\tilde{q}$. It turns out that $\partial : C_{i+1}(\tilde{W}, \tilde{M}_0, f, v) \to C_i(\tilde{W}, \tilde{M}_0, f, v)$ is given by

$$\partial(p) = \sum_{q, \operatorname{ind} q = i} [p:q]\, q \tag{1}$$

when we identify p with the homology class of $W^s(\tilde{p}, \tilde{v})$ in $H_{i+1}(\tilde{W}_{i+1}, \tilde{W}_i)$, see Milnor [9, §7].

Definition 2.2. The *Morse-Smale complex* of the Morse function f with transverse f-gradient v is given by

$$C_i^{MS}(\tilde{W}, \tilde{M}_0, f, v) = \bigoplus_{p \in \mathrm{crit}_i(f)} \mathbb{Z}G$$

and the boundary is given by (1).

This gives an abstract definition of the Morse-Smale complex in terms of critical points and trajectories between them, but the description using relative homology groups will remain useful. Let us modify the filtrations for this.

Since M_0 and M_1 can be nonempty, the flow of $-v$ need not be defined on all of $W \times \mathbb{R}$, but we can use it to define a continuous map $\Theta : W \times \mathbb{R} \to W$ which agrees with the flow where the flow is defined. To do this we can put a collar on W to get a slightly bigger manifold W_c and extend $-v$ to W_c such that the new vector field vanishes on ∂W_c. The flow of the new vector field is now defined on $W_c \times \mathbb{R}$ and we get Θ by retracting W_c back to W. In particular we get $\Theta(x, t) = x$ for $(x, t) \in M_0 \times [0, \infty) \cup M_1 \times (-\infty, 0]$. For $t \in \mathbb{R}$ set $\Theta_t(x) = \Theta(x, t)$ and define

$$W_i^t = \begin{cases} \Theta_t(W_i) & \text{if } t \geq 0 \\ \bigcup_{0 \geq s \geq t} \Theta_s(W_i) & \text{if } t < 0 \end{cases}$$

Then $(W_i^t)_{i=-1}^n$ is also a filtration and clearly $H_i(\tilde{W}_i^t, \tilde{W}_{i-1}^t) \cong H_i(\tilde{W}_i, \tilde{W}_{i-1})$ where the isomorphism is induced by inclusion.

Note that for $t > s$ we have $W_i^t \subset W_i^s$ and for very large t W_i^t mainly consists of M_0 and the stable manifolds of critical points with index $\leq i$. Also for very negative s W_i^s is mainly W minus the unstable manifolds of critical points with index $\geq i + 1$. To make this more precise note that for $t > s$ we get a direct system $j_{ts} = j_* : H_*(\tilde{W}_i^t, \tilde{W}_{i-1}^t) \to H_i(\tilde{W}_i^s, \tilde{W}_{i-1}^s)$ consisting of isomorphisms and commuting diagrams

$$\begin{array}{ccc} H_i(\tilde{W}_i^t, \tilde{W}_{i-1}^t) & \xrightarrow{\partial_i} & H_{i-1}(\tilde{W}_{i-1}^t, \tilde{W}_{i-2}^t) \\ \Big\downarrow j_{ts} & & \Big\downarrow j_{ts} \\ H_i(\tilde{W}_i^s, \tilde{W}_{i-1}^s) & \xrightarrow{\partial_i} & H_{i-1}(\tilde{W}_{i-1}^s, \tilde{W}_{i-2}^s) \end{array}$$

Let us define

$$C_i = C_i(v) = W - \bigcup_{p, \mathrm{ind} p \geq i+1} W^u(p, v).$$

It is easy to see that $H_i(\tilde{C}_i, \tilde{C}_{i-1})$ is the direct limit of the above direct system and since all j_{ts} are isomorphisms we can describe the Morse-Smale complex in terms of this filtration.

Definition 2.3. Let Δ be a smooth triangulation of W which contains M_0 and M_1 as subcomplexes. We say Δ is *adjusted to* v, if every i-simplex σ^i intersects the unstable manifolds $W^u(p, v)$ transversely for all critical points p with $\mathrm{ind}\, p \geq i$.

To see the existence of adjusted triangulations, let $\psi : W \to W$ be a diffeomorphism homotopic to the identity and Δ a smooth triangulation of W. Then $\psi\Delta$ is the triangulation of W where simplices are composed with ψ. Then the corresponding chain complexes can be identified by choosing a lifting $\tilde{\psi} : \tilde{W} \to \tilde{W}$. homotopic to the identity.

So let Δ be any smooth triangulation and $\psi_{-1} = \mathrm{id}_W$. We can adjust ψ_{-1} near the 0-skeleton so that 0-simplices intersect all unstable manifolds transversely. Since the boundary of W is transverse to the flow, we can leave it invariant. This way we get a diffeomorphism ψ_0 isotopic to the identity. Now assume ψ_{k-1} is isotopic to the identity and every j-simplex of $\psi_{k-1}\Delta$ with $j \le k-1$ intersects the unstable manifolds transversely for critical points with index $\ge k-1$. We modify ψ_{k-1} on the k-skeleton so that k-simplices intersect $W^u(p,v)$ transversely for all p with index $\ge k$. Notice that for a k-simplex of $\psi_{k-1}\Delta$ this is already true for ψ_{k-1} near the boundary so we can leave the $(k-1)$-skeleton fixed. This way we obtain ψ_k isotopic to the identity and we can proceed by induction.

Then $\psi_{n-1}\Delta$ is adjusted to v. Furthermore we can find an adjusted triangulation $\psi\Delta$ with ψ as close as we like to the identity. Moreover, compactness gives that if Δ is adjusted to v, so is $\psi\Delta$ for every ψ close enough to the identity.

Denote the i-skeleton of a triangulation by $W^{(i)}$. If Δ is adjusted to v, we get that $W^{(i)} \subset C_i$. In particular the inclusion induces a map on homology $j_* : H_*(\tilde{W}^{(i)}, \tilde{W}^{(i-1)}) \to H_*(\tilde{C}_i, \tilde{C}_{i-1})$ which defines a chain map

$$\varphi_v : C_*^\Delta(\tilde{W}, \tilde{M}_0) \to C_*^{MS}(\tilde{W}, \tilde{M}_0, f, v).$$

If $\operatorname{ind} p = \dim \sigma$, we have that $\sigma \cap W^u(p,v)$ is a finite set by the transversality assumption. For the definition of $C_*^\Delta(\tilde{W}, \tilde{M}_0)$ as generated by simplices, every simplex has a chosen lift and an orientation. Using the coorientation of $W^u(p,v)$ we get an intersection number $[\sigma : p] \in \mathbb{Z}G$ such that

$$\varphi_v(\sigma) = \sum_{p, \operatorname{ind} p = \dim \sigma} [\sigma : p]\, p,$$

so φ_v can be expressed in terms of v only.

If we are given an arbitrary smooth triangulation of the cobordism W, the above construction shows how to modify the triangulation to get a chain map. Different maps ψ_1, ψ_2 can lead to different chain maps, but the chain homotopy type is well defined:

Lemma 2.4. *The chain maps induced by $\psi_1\Delta$ and $\psi_2\Delta$ are chain homotopic.*

Proof. Let $H' : W \times I \to W$ be a homotopy between ψ_1 and ψ_2. As above we can change H' to a homotopy $H : W \times I \to W$ between ψ_1 and ψ_2 such that $H(\sigma \times I)$ intersects $W^u(p,v)$ transversely for all critical points p with $\mathrm{ind} p \ge \dim \sigma + 1$. Lift H to a homotopy $\tilde{H} : \tilde{W} \times I \to \tilde{W}$ such that $\tilde{H}_0 = \tilde{\psi}_1$ is homotopic to the identity. Then define $H_i : C_i^\Delta(\tilde{W}, \tilde{M}_0) \to C_{i+1}^{MS}(\tilde{W}, \tilde{M}_0, f, v)$ by $H_i(\sigma) = (-1)^i \tilde{H}_*[\tilde{\sigma} \times I] \in H_{i+1}(C_{i+1}, C_i)$. Then

$$\begin{aligned} \partial H + H\partial(\sigma) &= (-1)^i \tilde{H}_* \partial[\tilde{\sigma} \times I] + (-1)^{i-1} \tilde{H}_*[\partial\tilde{\sigma} \times I] \\ &= \tilde{H}_*[\tilde{\sigma} \times 1] - \tilde{H}_*[\tilde{\sigma} \times 0] \\ &= \tilde{\psi}_2[\tilde{\sigma}] - \tilde{\psi}_1[\tilde{\sigma}], \end{aligned}$$

so H_i is the desired chain homotopy. ☐

We already know that the triangulated and the Morse-Smale complex have the same homology and now we show that φ_v is indeed a chain homotopy equivalence.

Theorem 2.5. *Let $f : W \to [a,b]$ be a Morse function, v a transverse f-gradient, Δ a triangulation adjusted to v and $p : \tilde{W} \to W$ a regular covering space. Then $\varphi_v : C_*^{\Delta}(\tilde{W}, \tilde{M}_0) \to C_*^{MS}(\tilde{W}, \tilde{M}_0, f, v)$ is a simple homotopy equivalence.*

Proof. Let Δ' be a subdivision of Δ. If $\psi\Delta'$ is adjusted to v, so is $\psi\Delta$. Moreover, the diagram

$$\begin{tikzcd} C_*^{\psi\Delta}(\tilde{W}, \tilde{M}_0) \arrow[rr, "\mathrm{sd}"] \arrow[dr, "\varphi_v"'] & & C_*^{\psi\Delta'}(\tilde{W}, \tilde{M}_0) \arrow[dl, "\varphi_v"] \\ & C_*^{MS}(\tilde{W}, \tilde{M}_0) & \end{tikzcd}$$

commutes, where sd is subdivision, a simple homotopy equivalence. By Munkres [10, §10] it is good enough to show the theorem for a special smooth triangulation. Recall the filtration $(W_i)_{i=-1}^n$ by compact cobordisms. Choose a triangulation such that each W_i is a subcomplex for all $-1 \leq i \leq n$ and so that for each critical point p of index i the disc $D_i(p) = W^s(p,v) \cap (W_i - \operatorname{int} W_{i-1})$ is a subcomplex. We set for $0 \leq k \leq n$ $C_*^{(k)} = C_*^{\Delta}(\tilde{W}_k, \tilde{M}_0)$. The complex $D_*^{(k)}$ is given by

$$D_i^{(k)} = \begin{cases} C_i^{MS}(\tilde{W}, \tilde{M}_0) & i \leq k \\ 0 & \text{otherwise} \end{cases}$$

The chain map φ_v induces maps $\varphi^{(k)} : C_*^{(k)} \to D_*^{(k)}$ and $\varphi^{(k,k-1)} : C_*^{(k)}/C_*^{(k-1)} \to D_*^{(k)}/D_*^{(k-1)}$. Since the diagram

$$\begin{tikzcd} 0 \arrow[r] & C_*^{(k-1)} \arrow[r] \arrow[d, "\varphi^{(k-1)}"] & C_*^{(k)} \arrow[r] \arrow[d, "\varphi^{(k)}"] & C_*^{(k)}/C_*^{(k-1)} \arrow[r] \arrow[d, "\varphi^{(k,k-1)}"] & 0 \\ 0 \arrow[r] & D_*^{(k-1)} \arrow[r] & D_*^{(k)} \arrow[r] & D_*^{(k)}/D_*^{(k-1)} \arrow[r] & 0 \end{tikzcd}$$

commutes, it suffices to show that each $\varphi^{(k,k-1)}$ is a simple homotopy equivalence to finish the proof.

Clearly $\varphi^{(k,k-1)}$ induces an isomorphism in homology, so it remains to show that it is simple. We set $D_i = \bigcup\limits_{p \in \mathrm{crit}_i(f)} D_i(p)$. Then the inclusion $i : C_*^{\Delta}(\tilde{W}_{i-1} \cup \tilde{D}_i, \tilde{W}_{i-1}) \to C_*^{\Delta}(\tilde{W}_i, \tilde{W}_{i-1})$ is the inclusion of the core of the handles into the handles, hence a simple homotopy equivalence. Now $\varphi^{(k,k-1)} \circ i$ is a simple homotopy equivalence by Cohen [1, 18.3], since we can choose the lifts of D_i so that the matrices representing $\varphi^{(k,k-1)} \circ i$ and the boundary operators have only integer values. Therefore $\varphi^{(k,k-1)}$ is a simple homotopy equivalence. □

Now given another Morse function $g : W \to [0,1]$ and a transverse g-gradient w, let $\Phi : W \to W$ be isotopic to the identity such that $\Phi(W^s(q,v))$ intersects $W^u(p,w)$ transversely for all critical points q of f and p of g with $\operatorname{ind} q \leq \operatorname{ind} p$. The existence of such a Φ can be seen by adapting the proof of the existence of an adjusted triangulation. Let $(W_i(v))_{i=-1}^n$ be a filtration by compact cobordisms to give the Morse-Smale complex with respect to (f,v). By the transversality condition there is a $t > 0$ such that $W_i^t \subset C_i(w)$ for all i and hence a chain map

$$\psi_{v,w} : C_*^{MS}(\tilde{W}, \tilde{M}_0, f, v) \to C_*^{MS}(\tilde{W}, \tilde{M}_0, g, w)$$

induced by inclusion. For $\operatorname{ind} q = \operatorname{ind} p$ we get that $\Phi(W^s(q,v)) \cap W^u(p,w)$ is a finite set and using the orientations and coorientations we get intersection numbers $[q:p] \in \mathbb{Z}G$ such that

$$\psi_{v,w}(q) = \sum_{p \in \mathrm{crit}_{\operatorname{ind} q}(f)} [q:p]\, p.$$

A different choice of Φ will lead to a chain homotopic map, compare the proof of Lemma 2.4. The chain map $\psi_{v,w}$ is also a simple chain homotopy equivalence, more precisely we get

Proposition 2.6. *Let $f_0, f_1, f_2 : W \to [0,1]$ be Morse functions and let v_i be a transverse f_i-gradient for $i = 0,1,2$. Then*

(1) $\psi_{v_0,v_1} \circ \varphi_{v_0} \simeq \varphi_{v_1}$
(2) $\psi_{v_1,v_2} \circ \psi_{v_0,v_1} \simeq \psi_{v_0,v_2}$

Here '$\simeq$' means 'chain homotopic'.

Proof. 1. Notice that if Φ is isotopic to the identity, then $f_0 \circ \Phi$ is a Morse function and $d\Phi^{-1} \circ v_0 \circ \Phi$ is an $f_0 \circ \Phi$-gradient giving the same Morse-Smale complex as (f_0, v_0). Therefore we assume that Φ is the identity. We can also assume the triangulation Δ is adjusted to both v_0 and v_1.

There is a $t > 0$ such that $W_i^t(v_0) \subset C_i(v_1)$ for all i. Also there is an $s > 0$ such that $\Theta_s^{v_0}(W^{(i)}) \subset W_i^t(v_0)$ for all i. Now Θ^{v_0} gives a homotopy between $\Theta_s^{v_0}$ and the identity. Modify this homotopy away from the endpoints to get a homotopy $h : W \times I \to W$ such that $h(W^{(i)} \times I) \subset C_{i+1}(v_1)$. Now define $H : C_i^{\Delta}(\tilde{W}, \tilde{M}_0) \to C_{i+1}^{MS}(\tilde{W}, \tilde{M}_0, f_1, v_1)$ by sending σ to $(-1)^i \tilde{h}_*[\tilde{\sigma} \times I] \in H_{i+1}(\tilde{C}_{i+1}(v_1), \tilde{C}_i(v_1))$. Then

$$\begin{aligned} \partial H + H\partial(\sigma) &= \tilde{h}_*[\tilde{\sigma} \times 1] - \tilde{h}_*[\tilde{\sigma} \times 0] \\ &= \Theta_s^{v_0}[\tilde{\sigma}] - [\tilde{\sigma}] \\ &= \psi_{v_0,v_1} \circ \varphi_{v_0}(\sigma) - \varphi_{v_1}(\sigma) \end{aligned}$$

Notice that we evaluate in $H_i(\tilde{C}_i(v_1), \tilde{C}_{i-1}(v_1))$. Now $\Theta_s^{v_0}[\tilde{\sigma}]$ represents $\varphi_{v_0}(\sigma)$ in $H_i(\tilde{W}_i^t(v_0), \tilde{W}_{i-1}^t(v_0)$ and passing to $H_i(\tilde{C}_i(v_1), \tilde{C}_{i-1}(v_1))$ is applying ψ_{v_0,v_1}.

To prove 2. let us again assume that Φ can be chosen the identity, so that we have

$$\begin{array}{lcl} W^s(q,v_0) & \pitchfork & W^u(p,v_1) \\ W^s(q,v_0) & \pitchfork & W^u(r,v_2) \\ W^s(p,v_1) & \pitchfork & W^u(r,v_2) \end{array}$$

for the relevant critical points.

Choose $t > 0$ such that $W_i^t(v_0) \subset C_i(v_1) \cup C_i(v_2)$ and $W_i^t(v_1) \subset C_i(v_2)$ for all i. Since $W_i^t(v_0)$ is a compact subset of $C_i(v_1)$, there is an $s > 0$ such that $\Theta_s^{v_1}(W_i^t(v_0)) \subset W_i^t(v_1)$ for all i. Let $h : W \times I \to W$ be a homotopy between the identity and $\Theta_s^{v_1}$ such that $h(W^s(p,v_0) \times I)$ intersects $W^u(r,v_2)$ transversely for $\operatorname{ind} p \leq \operatorname{ind} r - 1$. Using a lifting $\tilde{h}$ of h we get a chain homotopy $H : C_i^{MS}(\tilde{W}, \tilde{M}_0, f_0, v_0) = H_i(W_i^t(v_0), W_{i-1}^t(v_0)) \to H_{i+1}(C_{i+1}(v_2), C_i(v_2)) = C_{i+1}^{MS}(\tilde{W}, \tilde{M}_0, f_2, v_2)$ between $\psi_{v_1,v_2} \circ \psi_{v_0,v_1}$ and ψ_{v_0,v_2} as in 1. ∎

3. The Novikov complex of a circle valued Morse function

In this section we look at a closed connected smooth manifold M and smooth functions $f : M \to S^1$. If all critical points of f are nondegenerate, we call f a Morse function. Since locally we have a function to the reals we get an index for every critical point and there are only finitely many critical points. Let $\rho : \mathbb{R} \to S^1$ be the covering given by $\rho(x) = \exp(2\pi i x)$. If the homomorphism of the fundamental group $f_\# : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ is trivial, f can be lifted to a map to $\mathbb{R}$ and we are in the situation of Section 2. Therefore we assume that $f_\#$ is nontrivial. Then we can also assume that $f_\#$ is surjective since otherwise we can lift f to a finite covering $S^1 \to S^1$ such that the resulting map induces an epimorphism on fundamental group.

Let $\bar\rho : \bar M \to M$ be the connected infinite cyclic covering space corresponding to $\ker f_\#$. Then $f \circ \bar\rho$ lifts to a map $\bar f : \bar M \to \mathbb{R}$ and we can assume that $0 \in \mathbb{R}$ is a regular value. Set $N = \bar f^{-1}(\{0\})$, $M_N = \bar f^{-1}([0,1])$ and $N' = \bar f^{-1}(\{1\})$. Then $(M_N; N, N')$ is a cobordism and $\bar f$ restricts to a Morse function on this cobordism. Let $\rho_1 : \tilde M \to \bar M$ be a covering space of $\bar M$ with covering transformation group H such that $\tilde\rho = \bar\rho \circ \rho_1 : \tilde M \to M$ is a regular cover of M with covering transformation group G. We then write $\tilde f = \bar f \circ \rho_1 : \tilde M \to \mathbb{R}$ and $\tilde X = \rho_1^{-1}(X)$ for $X \subset \bar M$. Fix a covering transformation $t \in G$ such that $t(\tilde N') = \tilde N$. We have a short exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1$$

so that $G = H \times_\theta \mathbb{Z}$ is a semi direct product with $\theta : H \to H$ defined by $\theta(h) = t^{-1}ht$. We can identify the group ring $\mathbb{Z}G$ with the θ-twisted Laurent polynomial ring $\mathbb{Z}H_\theta[t, t^{-1}]$ where the elements are finite sums $\sum a_j t^j$ with $a_j \in \mathbb{Z}H$ and $at = t\theta(a)$ for $a \in \mathbb{Z}H$. By abuse of notation we use t both as an element of G and as the indeterminate in the polynomial ring.

Let α be the standard closed 1-form on S^1, i.e. such that $\rho^*\alpha = dx$.

Definition 3.1. A vector field v is called an *f-gradient*, if it is dual to $f^*\alpha$ with respect to some Riemannian metric. It is called *transverse*, if $W^s(p, v)$ and $W^u(q, v)$ intersect transversely for all critical points of f with $\operatorname{ind} p \le \operatorname{ind} q + 1$.

Standard transversality arguments show that transverse f-gradients form a generic set among all f-gradients, see e.g. Pajitnov [15].

For every critical point p of f choose orientations, respectively coorientations, of $W^s(p, v)$, respectively $W^u(p, v)$, as in Section 2. Also choose a lift $\tilde p \in \tilde M$ for every critical point and lift the orientations as well. The lift of v to $\bar M$ is denoted by $\bar v$ and the lift to $\tilde M$ by $\tilde v$.

Now define for $g_1, g_2 \in G$ and two critical points p, q with $\operatorname{ind} p = \operatorname{ind} q + 1$ $[g_1\tilde p : g_2\tilde q] \in \mathbb{Z}^G$ by $[g_1\tilde p : g_2\tilde q](g) = \sum \varepsilon(\tilde\gamma)$, where the sum is taken over the trajectories of $-\tilde v$ from $g_1\tilde p$ to $gg_2\tilde q$.

Lemma 3.2. *We have*

(1) $[g_1\tilde p : g_2\tilde q] = g_1[\tilde p : \tilde q]g_2^{-1}$.
(2) $[g_1\tilde p : g_2\tilde q]|_H \in \mathbb{Z}H$.

Proof. By definition $[g_1\tilde p : g_2\tilde q](g) = [g_1\tilde p : \tilde q](gg_2) = [g_1\tilde p : \tilde q] \cdot g_2^{-1}(g)$. If $\tilde\gamma$ is a trajectory from $g_1\tilde p$ to $gg_2\tilde q$, then $g_1^{-1}\tilde\gamma$ is a trajectory from $\tilde p$ to $g_1^{-1}gg_2\tilde q$ with $\varepsilon(g_1^{-1}\tilde\gamma) = \varepsilon(\tilde\gamma)$, so $[g_1\tilde p : g_2\tilde q](g) = [\tilde p : g_2\tilde q](g_1^{-1}g) = g_1 \cdot [\tilde p : g_2\tilde q](g)$.

2. There exist integers $i < j$ such that $g_1\tilde{p}, g_2\tilde{q} \in \tilde{f}^{-1}([i,j])$. Then $[g_1\tilde{p} : g_2 q]|_H$ is the coefficient between $\rho_1(g_1\tilde{p})$ and $\rho_1(g_2\tilde{q})$ in the boundary of the Morse-Smale complex belonging to the cobordism $\bar{f}^{-1}([i,j])$. □

Definition 3.3. The *Novikov ring* $\mathbb{Z}H_\theta((t)) = \mathbb{Z}H_\theta[[t]][t^{-1}]$ consists of formal power series $\sum_{j=-\infty}^{\infty} a_j t^j$ with $a_j \in \mathbb{Z}H$ and $\{j \leq 0 \,|\, a_j \neq 0\}$ is finite. Again $at = t\theta(a)$ for $a \in \mathbb{Z}H$.

The *Novikov complex* of the circle valued Morse function $f : M \to S^1$ with transverse f-gradient v is the finitely generated free $\mathbb{Z}H_\theta((t))$ complex $C_*(\tilde{M}, f, v)$ generated by the critical points of f and graded by the index. The boundary is given by

$$\partial_i(p) = \sum_{q \in \mathrm{crit}_{i-1}(f)} \sum_{j=-\infty}^{\infty} [\tilde{p} : t^j \tilde{q}]|_H \, t^j \, q. \tag{2}$$

Notice that for $\tilde{f}(t^j\tilde{q}) > \tilde{f}(\tilde{p})$ we get $[\tilde{p} : t^j\tilde{q}]|_H = 0$, so ∂ is a well defined module homomorphism. To see that $C_*(\tilde{M}, f, v)$ is indeed a chain complex we have to show that $\partial^2 = 0$. This is independent of the choice of liftings of critical points. For if we replace $\tilde{p}$ by $g\tilde{p}$ for some $g \in G$ to get a complex C', we can define $\varphi : C_*(\tilde{M}, f, v) \to C'_*$ by $\varphi(q) = q$ for $q \neq p$ and $\varphi(p) = g^{-1}p$. Then $\partial' = \varphi \circ \partial \circ \varphi^{-1}$.

So choose the liftings of the critical points in $\tilde{M}_N$. For nonnegative integers j let $M_j^1 = \bar{f}^{-1}([-j, 1])$ and $M_j = \bar{f}^{-1}(\{-j\})$. Also let $\mathbb{Z}H_\theta[[t]]$ be the subring of $\mathbb{Z}H_\theta((t))$ consisting of formal power series $\sum_{j=0}^{\infty} a_j t^j$. That the Novikov complex is a chain complex now follows from the next Lemma.

Lemma 3.4. *With the liftings chosen in* $\tilde{M}_N$ *we get that*

$$C_*(\tilde{M}, f, v) = \mathbb{Z}H_\theta((t)) \otimes_{\mathbb{Z}H_\theta[[t]]} \varprojlim C_*^{MS}(\tilde{M}_j^1, \tilde{M}_j, \bar{f}|, \bar{v}|)$$

where the inverse limit is a finitely generated free $\mathbb{Z}H_\theta[[t]]$ *complex generated by the critical points of* f.

Proof. For $i = 0, \ldots, n$ let

$$C_i(j) = \bar{f}^{-1}((-\infty, 1]) - \bigcup_{k=0}^{j} \bigcup_{p, \mathrm{ind}\, p \geq i+1} W^u(t^k p, \bar{v}).$$

Then $C_i^{MS}(\tilde{M}_j^1, \tilde{M}_j, \bar{f}|, \bar{v}|) = H_i(\tilde{C}_i(j), \tilde{C}_{i-1}(j))$ is a free $\mathbb{Z}H$ module generated by the critical points of $\bar{f}|_{M_j^1}$ having index i. The inclusion $M_i(j+1) \subset M_i(j)$ gives the inverse system of Morse-Smale complexes. It is now easy to see that the inverse limit is a free $\mathbb{Z}H_\theta[[t]]$ chain complex generated by the critical points of f such that the boundary is given by (2), hence the result. □

The inverse limit description is very useful in determining the chain homotopy type of the Novikov complex. Let Δ be a smooth triangulation of M, the term *adjusted to* v carries over from the real valued case. The triangulation lifts to triangulations of $\bar{M}$ and $\tilde{M}$. By Section 2 we can find a triangulation adjusted to a compact cobordism $\bar{f}^{-1}([i,j])$ and get a certain openness and density result. Hence we can find a generic set of adjusted triangulations.

As in Section 2 we can now define intersection numbers $[g_1\tilde{\sigma} : g_2\tilde{p}] \in \mathbb{Z}G$ such that

$[g_1\tilde{\sigma} : g_2\tilde{p}](g)$ is the signed number of points in $g_1\tilde{\sigma} \cap W^u(gg_2\tilde{p}, \tilde{v})$. Lemma 3.2 carries over and we define $\varphi_v : C^\Delta_*(\tilde{M}) \to C_*(\tilde{M}, f, v)$ by

$$\varphi_v(\sigma) = \sum_{p, \operatorname{ind} p = \dim \sigma} \sum_{j=-\infty}^{\infty} [\tilde{\sigma} : t^j \tilde{p}]|_H \, t^j \, p.$$

Proposition 3.5. *φ_v is a chain homotopy equivalence.*

Proof. Since $\varphi_v(\sigma) = \varprojlim l_* : H_i(\tilde{\sigma}, \partial\tilde{\sigma}) \to H_i(\tilde{C}_i(j), \tilde{C}_{i-1}(j))([\tilde{\sigma}])$, it is a chain map. Lemma 2.4 carries over. If the triangulation contains $N = f^{-1}(\{1\})$ as a subcomplex, we also get commutative diagrams

$$\begin{array}{ccc} C^\Delta_*(\tilde{M}^1_j, \tilde{M}_j) & \longleftarrow & C^\Delta_*(\tilde{M}^1_{j+1}, \tilde{M}_{j+1}) \\ \Big\downarrow \varphi^j_v & & \Big\downarrow \varphi^{j+1}_v \\ C^{MS}_*(\tilde{M}^1_j, \tilde{M}_j, \bar{f}|, \bar{v}|) & \longleftarrow & C^{MS}_*(\tilde{M}^1_{j+1}, \tilde{M}_{j+1}, \bar{f}|, \bar{v}|) \end{array}$$

and $\varphi_v = \mathrm{id} \otimes \varprojlim \varphi^j_v$. Since the φ^j_v are chain homotopy equivalences by Theorem 2.5, so is the inverse limit as a chain map between finitely generated free chain complexes inducing an isomorphism on homology. Therefore φ_v is a chain homotopy equivalence. □

Let us also define the chain maps $\psi_{v,w} : C_*(\tilde{M}, f, v) \to C_*(\tilde{M}, g, w)$ where $g : M \to S^{-1}$ is another Morse function homotopic to f and w a transverse g-gradient. As in Section 2 we can assume, after possibly altering g with a map isotopic to the identity on M, that $W^s(q, v)$ intersects $W^u(p, w)$ transversely for all critical points q of f and p of g with $\operatorname{ind} q \leq \operatorname{ind} p$. For $\operatorname{ind} q = \operatorname{ind} p$ and $g_1, g_2 \in G$ define $[g_1\tilde{q} : g_2\tilde{p}] \in \mathbb{Z}^G$ such that $[g_1\tilde{q} : g_2\tilde{p}](g)$ is the signed number of points in $W^s(g_1\tilde{q}, \tilde{v}) \cap W^u(gg_2\tilde{p}, \tilde{w})$. Then we define

$$\psi_{v,w}(q) = \sum_{p \in \mathrm{crit}_{\operatorname{ind} q}(g)} \sum_{j=-\infty}^{\infty} [\tilde{q} : t^j \tilde{p}]|_H \, t^j \, p.$$

Proposition 3.6. *Let $f_i : M \to S^1$ be homotopic Morse functions for and v_i be transverse f_i-gradients for $i = 0, 1, 2$. Then*

(1) $\psi_{v_0,v_1} \circ \varphi_{v_0} \simeq \varphi_{v_1}$.
(2) $\psi_{v_1,v_2} \circ \psi_{v_0,v_1} \simeq \psi_{v_0,v_2}$.

Proof. We show 2. using Proposition 2.6, 1. will follow analogously. We assume the same transversality assumptions as in the proof of Proposition 2.6.

Let $\bar{f}_i : \bar{M} \to \mathbb{R}$ be liftings of the f_i such that $0 \in \mathbb{R}$ is a regular value for all of them. There are integers l, m such that $\bar{f}_1(\bar{f}_0^{-1}((-\infty, 0])) \subset (-\infty, l]$ and $\bar{f}_2(\bar{f}_1^{-1}((-\infty, l])) \subset (-\infty, m]$. Let $k_0 = 0$, $k_1 = l$ and $k_2 = m$. For $i = 0, 1, 2$ define $M^i_j = \bar{f}_i^{-1}([-j + k_i, 1 + k_i])$, $N^i_j = \bar{f}_i^{-1}(\{-j + k_i\})$ and $M^i_\infty = \bar{f}_i^{-1}((-\infty, 1 + k_i])$. By using thin enough filtrations we get for $0 \leq i_1 < i_2 \leq 2$ chain maps $\psi^j_{i_1 i_2} : C^{MS}_*(\tilde{M}^{i_1}_j, \tilde{N}^{i_1}_j, \bar{f}_{i_1}|, \bar{v}_{i_1}|) \to C^{MS}_*(\tilde{M}^{i_2}_j, \tilde{N}^{i_2}_j, \bar{f}_{i_2}|, \bar{v}_{i_2}|)$ induced by inclusion as in the proof of Proposition 2.6. We also get a chain homotopy $H^j : \psi^j_{12} \circ \psi^j_{01} \simeq \psi^j_{02}$ which is induced by a homotopy $h^j : M^0_\infty \times I \to M^2_\infty$ between the inclusion $k_{02} : M^0_\infty \to M^2_\infty$ and $k_{12} \circ \Theta^{\bar{v}_1}_{s_j} \circ k_{01}$. Here $\Theta^{\bar{v}_1}$ is the flow of $-\bar{v}_1$ and the time s_j is chosen so that the thin filtration of M^0_j flows into the thin filtration of M^1_j. Also $h^j(W^s(\bar{p}, \bar{v}_0) \times I)$

intersects $W^u(\bar r, \bar v_2)$ transversely for $\operatorname{ind} \bar p \leq \operatorname{ind} \bar r - 1$ where $\bar p \in M_j^0$ and $\bar r \in M_j^2$.

To define H^{j+1} we need a new homotopy h^{j+1}. To get this use h^j and flow a little longer along the flow of $-\bar v_1$ until the thin filtration of M_{j+1}^0 includes into the thin filtration of M_{j+1}^1. Then we change this homotopy to get a new homotopy h^{j+1} such that $h^{j+1}(W^s(\bar p, \bar v_0) \times I)$ intersects $W^u(\bar r, \bar v_2)$ transversely for $\operatorname{ind} \bar p \leq \operatorname{ind} \bar r - 1$ where $\bar p \in M_{j+1}^0$ and $\bar r \in M_{j+1}^2$. Since this was already true for $\bar p \in M_j^0$ and $\bar r \in M_j^2$ we can make the changes so that the diagram

$$\begin{array}{ccc} C_*^{MS}(\tilde M_j^0, \tilde N_j^0, \bar f_0|, \bar v_0|) & \longleftarrow & C_*^{MS}(\tilde M_{j+1}^0, \tilde N_{j+1}^0, \bar f_0|, \bar v_0|) \\ \Big\downarrow H^j & & \Big\downarrow H^{j+1} \\ C_*^{MS}(\tilde M_j^2, \tilde N_j^2, \bar f_2|, \bar v_2|) & \longleftarrow & C_*^{MS}(\tilde M_{j+1}^2, \tilde N_{j+1}^2, \bar f_2|, \bar v_2|) \end{array}$$

commutes. The inverse limit of H^j is a chain homotopy between $\varprojlim \psi_{12}^j \circ \psi_{01}^j$ and $\varprojlim \psi_{02}^j$. But $\psi_{v_{i_1}, v_{i_2}} = \operatorname{id} \otimes \varprojlim \psi_{i_1 i_2}^j$, if the liftings of the critical points of f_i are chosen in $\tilde f^{-1}([k_i, k_i + 1])$. Since the ψ_{v_0, v_1} behave well with respect to change of basis we get the result. $\square$

Remark 3.7. The chain homotopy H constructed in the proof can be written as

$$H(p) = \sum_{r \in \operatorname{crit}_{\operatorname{ind} p + 1}(f_2)} \sum_{j=-\infty}^{\infty} [p : t^j r]\, t^j\, r.$$

To get $[p : t^j r](h) \neq 0$, there exists a trajectory of $-\tilde v_0$ from $\tilde p$ to a point $\tilde x$, a trajectory of $-\tilde v_1$ from $\tilde x$ to a point $\tilde y$ and a trajectory of $-\tilde v_2$ from $\tilde y$ to $h t^j \tilde r$. This observation will become useful in Section 4.

4. The Novikov complex of a Morse closed 1-form

Now we want to look at closed 1-forms on a closed connected smooth manifold M. A closed 1-form ω induces a homomorphism $\xi_{[\omega]} = \xi : \pi_1(M) \to \mathbb{R}$ by $\xi([\gamma]) = \int_\gamma \omega$, where γ is a smooth representative of $[\gamma] \in \pi_1(M)$. By de Rham's theorem all homomorphisms $\xi : \pi_1(M) \to \mathbb{R}$ arise in such a way.

A closed 1-form is locally exact and we say that ω is a *Morse form*, if locally the functions $f : U \to \mathbb{R}$ with $df = \omega|_U$ have nondegenerate critical points only. Again we get that a Morse form has only finitely many critical points, each with an index.

Since $\pi_1(M)$ is finitely presented, $\operatorname{im} \xi$ is a finitely generated subgroup of $\mathbb{R}$, hence $\mathbb{Z}^k$ for some $k \geq 0$. If $k = 0$, ω is the differential of a smooth function $f : M \to \mathbb{R}$ and we are in the situation of Section 2. If $k = 1$ we call ω *rational* and if $k > 1$ we call ω *irrational*.

There is a minimal regular covering space $\bar\rho : \bar M \to M$ such that $\bar\rho^* \omega = d\bar f$ for some $\bar f : \bar M \to \mathbb{R}$, namely the one corresponding to $\ker \xi$. The group of covering transformations is $\mathbb{Z}^k$. If $k = 1$ let $t \in \mathbb{Z}$ be the generator such that $b = \bar f(\bar x) - \bar f(t\bar x) > 0$ for all $\bar x \in \bar M$. Then we can define $f : M \to S^1$ by $f(x) = \exp(2\pi i \bar f(\bar x)/b)$, where $\bar x \in \bar M$ is a lift of $x \in M$. Then $\omega = b f^* \alpha$ with α the standard volume form on S^1 and we see that rational Morse forms correspond to circle valued Morse functions.

Definition 4.1. Let ω be a Morse form. A vector field v is called an *ω-gradient*, if it is dual to ω with respect to some Riemannian metric. It is called *transverse*, if $W^s(p,v)$ and $W^u(q,v)$ intersect transversely for all critical points of ω with $\operatorname{ind} p \leq \operatorname{ind} q + 1$.

The following is an easy lemma proven in [21].

Lemma 4.2. *Let ω be a Morse form and v a vector field. Then v is an ω-gradient if and only if*

1. *For every critical point p of ω there exists a neighborhood U_p of p and a Riemannian metric g on U_p such that $\omega_x(X) = g(X, v(x))$ for every $x \in U_p$ and $X \in T_xU_p$.*
2. *If $\omega_x \neq 0$, then $\omega_x(v(x)) > 0$.*

We want to define a Novikov complex for a pair (ω, v) but we need the right ring for this. So for a group G and a homomorphism $\xi : G \to \mathbb{R}$ let

$$\widehat{\mathbb{Z}G}_\xi = \{\lambda \in \mathbb{Z}^G \,|\, \forall R \in \mathbb{R}\ \#\{g \in G \,|\, \lambda(g) \neq 0 \text{ and } \xi(g) \geq R\} < \infty\}.$$

Clearly $\mathbb{Z}G \subset \widehat{\mathbb{Z}G}_\xi$ and the multiplication of $\mathbb{Z}G$ extends to $\widehat{\mathbb{Z}G}_\xi$. Then $\widehat{\mathbb{Z}G}_\xi$ becomes a ring, the *Novikov ring*.

For a surjective homomorphism $\xi : G \to \mathbb{Z}$ we want to show that this coincides with the Novikov ring given in Section 3. Let $t \in G$ satisfy $\xi(t) = -1$ and let $H = \ker \xi$. Let $\theta : H \to H$ be $\theta(h) = t^{-1}ht$. For every $g \in G$ there is a unique $n_g \in \mathbb{Z}$ and $h_g \in H$ such that $g = h_g \cdot t^{n_g}$. So for $a \in \mathbb{Z}H$ and $g \in G$ define $at^j \in \mathbb{Z}G$ by $at^j(g) = a(h_g) \cdot t^j(t^{n_g})$. It is easy to see that this induces a ring isomorphism between $\mathbb{Z}H_\theta((t))$ and $\widehat{\mathbb{Z}G}_\xi$.

Given a Morse form ω and a transverse ω-gradient we now want to define the Novikov complex $C_*(\tilde{M}, \omega, v)$ as before. For this we need the intersection numbers $[\tilde{p} : \tilde{q}]$ to be elements of $\widehat{\mathbb{Z}G}_\xi$. We achieve this by an approximation result.

Lemma 4.3. *Let ω be a Morse form and v an ω-gradient. There is a rational Morse form ω' such that v is also an ω'-gradient, ω' agrees with ω in a neighborhood of the critical points of ω and the homomorphism $\xi_{[\omega']}$ vanishes on* $\ker \xi_{[\omega]}$.

Proof. Let $g_1, \ldots, g_k \in G$ be a minimal set of generators of $\operatorname{im} \xi_{[\omega]}$ and let ω_i be closed 1-forms for $i = 1, \ldots, k$ such that $\xi_i = \xi_{[\omega_i]} : G \to \mathbb{Z}$ satisfies $\xi_i(g_j) = \delta_{ij}$ and all ξ_i vanish on $\ker \xi_{[\omega]}$. We can assume that the ω_i vanish in a neighborhood of the critical points of ω. Let $\varepsilon \in \mathbb{R}^k$. For $\|\varepsilon\|$ small enough the closed 1-form $\omega_\varepsilon = \omega + \sum_{i=1}^k \varepsilon_i \omega_i$ will have the property that v is an ω_ε-gradient by Lemma 4.2. Now choose the ε_i so that $\xi_{[\omega_\varepsilon]}$ factors through $\mathbb{Q}$. $\square$

So if ω is a Morse form, v a transverse ω-gradient, $\tilde{\rho} : \tilde{M} \to M$ a regular covering space with covering transformation group G factoring through $\bar{M}$ we can define a Novikov complex $C_*(\tilde{M}, \omega', v)$ over $\widehat{\mathbb{Z}G}_{\xi'}$ using ω' from Lemma 4.3 and writing $\xi' = \xi_{[\omega']}$. The next lemma shows that this complex pulls back to $\widehat{\mathbb{Z}G}_{\xi'} \cap \widehat{\mathbb{Z}G}_\xi \subset \mathbb{Z}^G$, a subring of both $\widehat{\mathbb{Z}G}_{\xi'}$ and $\widehat{\mathbb{Z}G}_\xi$.

Lemma 4.4. *Let ω_1, ω_2 be Morse forms that agree near the common set of critical points with corresponding homomorphisms $\xi_1\, \xi_2 : G \to \mathbb{R}$. Let v be both an ω_1- and ω_2-gradient. Then there exist constants A, $B \in \mathbb{R}$ with $A > 0$ such that whenever* *there is $g \in G$ and a trajectory $\tilde{\gamma}$ of $-\tilde{v}$ between the critical points $g\tilde{q}$ and $\tilde{p}$, then $\xi_1(g) \le A\xi_2(g) + B$.*

Proof. For every pair of critical points p, q of ω_i we can choose a path $\tilde{\gamma}_{pq}$ in $\tilde{M}$ from $\tilde{p}$ to $\tilde{q}$. There is a $K > 0$ such that $|\int_{\gamma_{pq}} \omega_i| \le K$ for $i = 1, 2$ and all pairs of critical points. Since ω_1 and ω_2 agree near the critical points there exists a $C \in (0, 1)$ such that $\omega_1(v(x)) \ge C\omega_2(v(x))$ for all $x \in M$ by compactness. Now let $g \in G$ be as in the statement. Then

$$\begin{aligned}
\xi_1(g) &= \int_{\gamma_{qp}} \omega_1 + \int_{\gamma} \omega_1 \;\le\; K - \int_{-\infty}^{\infty} \omega_1(v(\gamma(t)))\,dt \\
&\le K - C\int_{-\infty}^{\infty} \omega_2(v(\gamma(t)))\,dt \\
&\le K + CK + C\int_{\gamma_{qp}} \omega_2 + C\int_{\gamma} \omega_2 \;=\; K(1+C) + C\xi_2(g),
\end{aligned}$$

which gives the result. □

Corollary 4.5. *Let ω be a Morse form, v a transverse ω-gradient and ω' as in Lemma 4.3. Let p, q be critical points of ω with $\operatorname{ind} p = \operatorname{ind} q + 1$. Then $[\tilde{p} : \tilde{q}] \in \widehat{\mathbb{Z}G}_{\xi'} \cap \widehat{\mathbb{Z}G}_{\xi}$.*

Definition 4.6. The *Novikov complex* of the Morse form ω and the transverse ω-gradient v is the free $\widehat{\mathbb{Z}G}_{\xi}$ complex $C_*(\tilde{M}, \omega, v)$ generated by the critical points of ω and graded by the index. The boundary is given by

$$\partial_i(p) = \sum_{q \in \mathrm{crit}_{i-1}\omega} [\tilde{p} : \tilde{q}]\, q. \tag{3}$$

Since $[\tilde{p} : \tilde{q}] \in \widehat{\mathbb{Z}G}_{\xi'} \cap \widehat{\mathbb{Z}G}_{\xi}$, we can define a graded module C_* over $\widehat{\mathbb{Z}G}_{\xi'} \cap \widehat{\mathbb{Z}G}_{\xi}$ and a homomorphism ∂ given by (3). Tensoring with $\widehat{\mathbb{Z}G}_{\xi'}$ gives the Novikov complex of Section 3, but since $\widehat{\mathbb{Z}G}_{\xi'} \cap \widehat{\mathbb{Z}G}_{\xi}$ includes into $\widehat{\mathbb{Z}G}_{\xi'}$, C_* is already a chain complex. Now $C_*(\tilde{M}, \omega, v) = \widehat{\mathbb{Z}G}_{\xi} \otimes_{\widehat{\mathbb{Z}G}_{\xi'} \cap \widehat{\mathbb{Z}G}_{\xi}} C_*$.

To identify the chain homotopy type we need the chain homotopy equivalences $\varphi_v : \widehat{\mathbb{Z}G}_{\xi} \otimes_{\mathbb{Z}G} C_*^{\Delta}(\tilde{M}) \to C_*(\tilde{M}, \omega, v)$ and $\psi_{v,w} : C_*(\tilde{M}, \omega_1, v) \to C_*(\tilde{M}, \omega_2, w)$ already known in the rational case. To get them as chain maps, we merely have to show that the intersection numbers $[\tilde{\sigma} : \tilde{p}]$ and $[\tilde{p} : \tilde{q}]$ as defined in Section 3 lie in $\widehat{\mathbb{Z}G}_{\xi'} \cap \widehat{\mathbb{Z}G}_{\xi}$ as well.

Proposition 4.7. *Let ω_0, ω_1, ω_2 be cohomologous Morse forms and for $i = 0, 1, 2$ let v_i be a transverse ω_i-gradient. Then*

(1) $\psi_{v_0,v_1} \circ \varphi_{v_0} \simeq \varphi_{v_1}$.
(2) $\psi_{v_1,v_2} \circ \psi_{v_0,v_1} \simeq \psi_{v_0,v_2}$.

Proof. Recall the proof of Lemma 4.3. By choosing the closed 1-forms to vanish in a neighborhood of all critical points of ω_0, ω_1, ω_2 we can find rational approximations ω_i' for ω_i such that ω_0', ω_1' and ω_2' are cohomologous. In fact we get $\omega_i - \omega_j = \omega_i' - \omega_j'$. With Proposition 3.6 we get the φ_{v_i}, ψ_{v_i,v_j} and the chain homotopies over $\widehat{\mathbb{Z}G}_{\xi'}$. It remains to show that all the coefficients lie in $\widehat{\mathbb{Z}G}_{\xi'} \cap \widehat{\mathbb{Z}G}_{\xi}$. This requires an argument similar to Lemma 4.4 and we give it for the chain homotopy $H : \psi_{v_1,v_2} \circ \psi_{v_0,v_1} \simeq$

ψ_{v_0,v_2}.

In Remark 3.7 we pointed out that a nonzero coefficient $[\tilde{p}:\tilde{r}]$ gives the existence of a trajectory $\tilde{\gamma}_0$ of $-\tilde{v}_0$ from $\tilde{p}$ to $\tilde{x}\in\tilde{M}$, a trajectory $\tilde{\gamma}_1$ of $-\tilde{v}_1$ from $\tilde{x}$ to $\tilde{y}\in\tilde{M}$ and a trajectory of $-\tilde{v}_2$ from $\tilde{y}$ to $g\tilde{r}$. As in Lemma 4.4 choose $K>0$ and paths $\tilde{\gamma}_{rp}$ between critical points $\tilde{r}$ of ω_2 and $\tilde{p}$ of ω_0 such that $|\int_{\gamma_{rp}}\omega_i|\leq K$. As in Lemma 4.4 there is a $C\in(0,1)$ with $\omega_i(v_i(x))\geq C\omega_i'(v_i(x))$ for all $i=0,1,2$ and $x\in M$. Let $df_{01}=\omega_0-\omega_1$ and $df_{02}=\omega_0-\omega_2$. Then

$$\begin{aligned}
\xi(g) &= \int_{\gamma_{rp}}\omega_0+\int_{\gamma_0}\omega_0+\int_{\gamma_1}\omega_0+\int_{\gamma_2}\omega_0\\
&= \int_{\gamma_{rp}}\omega_0+\int_{\gamma_0}\omega_0+\int_{\gamma_1}\omega_1+\int_{\gamma_2}\omega_2+\int_{\gamma_1}df_{01}+\int_{\gamma_2}df_{02}\\
&\leq K(1+C)+\int_{\gamma_{rp}}\omega_0'+C\int_{\gamma_0}\omega_0'+C\int_{\gamma_1}\omega_1'+C\int_{\gamma_2}\omega_2'+\int_{\gamma_1}df_{01}+\int_{\gamma_2}df_{02}\\
&= K(1+C)+(1-C)\left(\int_{\gamma_1}df_{01}+\int_{\gamma_2}df_{02}\right)+C\xi'(g).
\end{aligned}$$

Notice that $\int_{\gamma_i}df_{0i}$ is bounded by $\max\{f_{0i}(x)-f_{0i}(y)\,|\,x,y\in M\}$. Therefore $[\tilde{p}:\tilde{r}]\in\widehat{\mathbb{Z}G}_{\xi'}\cap\widehat{\mathbb{Z}G}_{\xi}$ and the result follows. □

Theorem 4.8. *Let ω_1,ω_2 be cohomologous Morse forms, v a transverse ω_1-gradient and w a transverse ω_2-gradient. Then $\psi_{v,w}:C_*(\tilde{M},\omega_1,v)\to C_*(\tilde{M},\omega_2,w)$ and $\varphi_v:\widehat{\mathbb{Z}G}_{\xi}\otimes_{\mathbb{Z}G}C_*^{\Delta}(\tilde{M})\to C_*(\omega_1,v)$ are chain homotopy equivalences.*

Proof. That $\psi_{v,w}$ is a chain homotopy equivalence follows directly from Proposition 4.7.2 since $\psi_{v,v}=\mathrm{id}$ by definition. By Proposition 4.7.1 it now suffices to show that φ_u is a chain homotopy equivalence for a particular ω_3 with transverse ω_3-gradient u. By a nice trick of Latour [8] there is a Morse form ω_3 cohomologous to ω_1 and a transverse ω_3-gradient u such that u is also a gradient of a Morse function $f:M\to\mathbb{R}$. Namely choose a Morse function $f:M\to\mathbb{R}$ and a transverse f-gradient u and change ω_1 to ω' in a contractible neighborhood of every critical point of f such that ω' is constant 0 near the critical points of f. This is possible since ω_1 is locally exact. Now for $C>0$ large enough $\omega_3=\omega'+C\cdot df$ has u as a gradient. Then $C_*(\tilde{M},\omega_3,u)=\widehat{\mathbb{Z}G}_{\xi}\otimes_{\mathbb{Z}G}C_*(\tilde{M},f,u)$ and $\varphi_u=\mathrm{id}\otimes\varphi_u^{MS}$. Since φ_u^{MS} is a chain homotopy equivalence, so is φ_u. □

5. Continuation

Another way to describe a chain homotopy equivalence is the principle of continuation coming out of Floer theory, see also Schwarz [24] or Poźniak [19]. To describe this we will follow the exposition of Poźniak [19, §2.6].

Let ω_0 and ω_1 be cohomologous Morse forms and v_i be a transverse ω_i-gradient for $i=0,1$. We have $\omega_0-\omega_1=df$ for some $f:M\to\mathbb{R}$. Define a Morse form Ω on $M\times[0,1]$ as follows. Let $v:[0,1]\to[0,1]$ be a smooth function constant 0 near 0 and constant 1 near 1. For $C>0$ let

$$\Omega=\omega_0-C\sin\pi t\,dt+d(f\cdot v)=\omega_0-C\sin\pi t\,dt+df\cdot v(t)+f\cdot v'(t)\,dt.$$

We can choose the $C>0$ so large that the dt summand only vanishes for $t=0,1$. At $t=0$ we get $\omega=\omega_0$ and at $t=1$ we get $\Omega=\omega_1$. In particular the critical points of Ω are $(p,0)$ with $p \in \operatorname{crit}\omega_0$ and $(q,1)$ with $q \in \operatorname{crit}\omega_1$. Also $\operatorname{ind}(p,0) = \operatorname{ind} p + 1$ and $\operatorname{ind}(q,1) = \operatorname{ind} q$. Now let v be a transverse Ω-gradient that agrees with v_i on $M \times \{i\}$ for $i = 0,1$. Then we can form the Novikov complex $C_* = C_*(\tilde{M} \times [0,1], \Omega, v)$ as before. To see that this is in fact a chain complex one can extend Ω to a Morse form on $M \times [0,2]/(x,0) = (x,2) = M \times S^1$ and get C_* to be a certain restriction of the corresponding Novikov complex, see Poźniak [19] for more details. Now it is clear that $C_i = C_i(\tilde{M}, \omega_1, v_1) \oplus C_{i-1}(\tilde{M}, \omega_0, v_0)$ and $C_*(\tilde{M}, \omega_1, v_1)$ is a subcomplex. Therefore C_* is the mapping cone of a chain map $c_{v_0,v_1} : C_*(\tilde{M}, \omega_0, v_0) \to C_*(\tilde{M}, \omega_1, v_1)$, the continuation map. It is obtained by counting the signed flowlines from $(p,0)$ to $(q,1)$. We show that c_{v_0,v_1} is chain homotopic to ψ_{v_0,v_1}.

Let $p_0 : C_* \to C_{*-1}(\tilde{M}, \omega_0, v_0)$ and $p_1 : C_* \to C_*(\tilde{M}, \omega_1, v_1)$ be the projections. Notice that p_1 is not a chain map, in fact $p_1\partial_i = \partial_i p_1 + (-1)^{i-1} c_{v_0,v_1} \circ p_0$.

Proposition 5.1. *Let Δ be a smooth triangulation of M. Then $c_{v_0,v_1} \circ \varphi_{v_0} \simeq \varphi_{v_1}$.*

Proof. We can assume that Δ is both adjusted to v_0 and v_1. We get a CW-structure $\Delta \times I$ of $M \times [0,1]$ by looking at the natural product structure. We want $\Delta \times I$ adjusted to v. Notice that for a critical point $(q,1)$ of Ω the unstable manifold $W^u((q,1),v)$ has one more dimension that $W^u(q,v_1)$ while $W^u((p,0),v) = W^u(p,v_0)$. So the simplices $\sigma \times \{1\}$ already intersect the $W^u((q,1),v)$ transversely if σ intersects $W^u(q,v_1)$ transversely. But the simplices $\sigma \times \{0\}$ do not intersect $W^u((p,0),v)$ transversely even if σ intersects $W^u(p,v_0)$ transversely. But we can extend Ω and v into a collar of $M \times \{0\}$ and push the triangulation of $M \times \{0\}$ into that collar. Now we can adjust $\Delta \times I$ to v such that the intersections of $W^u((p,0),v)$ and $\sigma \times I$ give the same coefficient as the intersection of σ with $W^u(p,v_0)$, i.e. we get $\varphi_{v_0}(\sigma) = p_0\varphi_v(\sigma \times I)$, where $\varphi_v : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^{\Delta \times I}(\tilde{M} \times I) \to C_*$ is the resulting chain map. We can also assume that $p_1\varphi_v(\sigma \times \{1\}) = \varphi_{v_1}(\sigma)$. Now define

$$H_i : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_i(\tilde{M}) \to C_{i+1}(\tilde{M}, \omega_1, v_1)$$

by $H_i(\sigma) = (-1)^i p_1 \circ \varphi_v(\sigma \times I)$. Then

$$\begin{aligned} \partial_{i+1}H_i + H_{i-1}\partial_i(\sigma) &= (-1)^i\partial_{i+1}p_1\varphi_v(\sigma \times I) + (-1)^{i-1}p_1\varphi_v(\partial\sigma \times I) \\ &= (-1)^i p_1\partial_{i+1}\varphi_v(\sigma \times I) - c_{v_0,v_1}p_0\varphi_v(\sigma \times I) + \\ &\quad +(-1)^{i-1}p_1\varphi_v(\partial\sigma \times I) \\ &= p_1\varphi_v(\sigma \times \{1\}) - p_1\varphi_v(\sigma \times \{0\}) - c_{v_0,v_1}\varphi_{v_0}(\sigma) \\ &= \varphi_{v_1}(\sigma) - c_{v_0,v_1}\varphi_{v_1}(\sigma). \end{aligned}$$

Notice that $p_1\varphi_v(\sigma \times \{0\}) = 0$, since $W^u((q,1),v) \cap M \times \{0\} = \emptyset$. □

Corollary 5.2. *Let ω_0 and ω_1 be cohomologous Morse forms and v_i be a transverse ω_i-gradient for $i = 0,1$. Then c_{v_0,v_1} is chain homotopic to ψ_{v_0,v_1}.*

Proof. Using Proposition 4.7, Theorem 4.8 and Proposition 5.1 we get

$$c_{v_0,v_1} \simeq \varphi_{v_1} \circ \varphi_{v_0}^{-1} \simeq \psi_{v_0,v_1},$$

hence the result. □

6. The simple homotopy type of the Novikov complex

In the exact case Theorem 2.5 already states that φ_v is a simple chain homotopy equivalence, by Proposition 2.6.1 the same is true for $\psi_{v,w}$. In the nonexact case the situation gets a little bit more complicated. First we need to decide what we mean by a simple chain homotopy equivalence.

If G is a group and $\xi : G \to \mathbb{R}$ a homomorphism, let $a \in \widehat{\mathbb{Z}G}_\xi$ satisfy $a(g) = 0$ for $g \in G$ with $\xi(g) \geq 0$. Then $1 - a \in \widehat{\mathbb{Z}G}_\xi$ is a unit with inverse $\sum_{k=0}^\infty a^k \in \widehat{\mathbb{Z}G}_\xi$. We call such units and units of the form $\pm g$ with $g \in G$ *trivial units*.

Definition 6.1. Let G be a group and $\xi : G \to \mathbb{R}$ a homomorphism.

(1) Let U be the subgroup of $K_1(\widehat{\mathbb{Z}G}_\xi)$ generated by the trivial units. Then $\mathrm{Wh}(G;\xi) := K_1(\widehat{\mathbb{Z}G}_\xi)/U$.
(2) A chain homotopy equivalence $\varphi : C_* \to D_*$ between finitely generated free based $\widehat{\mathbb{Z}G}_\xi$ complexes C_* and D_* is called *simple* , if $\tau(\varphi) = 0 \in \mathrm{Wh}(G;\xi)$.

In the rational case Pajitnov and Ranicki [18] obtain a decomposition result for $K_1(\widehat{\mathbb{Z}G}_\xi)$. In particular the natural map $i_* : \mathrm{Wh}(G) \to \mathrm{Wh}(G;\xi)$ is surjective then.

Pajitnov [12] in the circle valued case and Latour [8] in general already showed that there is a simple chain homotopy equivalence between $C_*(\omega, v)$ and $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C^\Delta_*(\tilde{M})$ and the purpose of this section is to show that φ_v is such an equivalence.

Theorem 6.2. *Let ω_1, ω_2 be cohomologous Morse forms, v a transverse ω_1-gradient and w a transverse ω_2-gradient. Then $\tau(\varphi_v) = \tau(\psi_{v,w}) = \tau(c_{v,w}) = 0 \in \mathrm{Wh}(G;\xi)$, i.e. all these chain homotopy equivalences are simple.*

Proof. We show that $\tau(\psi_{v,w}) = 0$. By Corollary 5.2 we then get $\tau(c_{v,w}) = 0$, since chain homotopic equivalences have the same torsion, see e.g. Ranicki [20, Prop.1.2]. To get the result for $\tau(\varphi_v)$ notice that by Proposition 4.7.1 it is enough to show $\tau(\varphi_u) = 0$ for a particular u. Recall Latour's trick in the proof of Theorem 4.8: we get $\tau(\varphi_u) = i_*\tau(\varphi_u^{MS})$ for the u constructed there. Here $i_* : \mathrm{Wh}(G) \to \mathrm{Wh}(G;\xi)$ is the natural map and $\tau(\varphi_u^{MS}) = 0$ by Theorem 2.5. Hence we get $\tau(\varphi_v) = 0$.

Since ω_1 and ω_2 are cohomologous, there is $f : M \to \mathbb{R}$ such that $\omega_2 = \omega_1 + df$. Let $v : [0,1] \to [0,1]$ be a smooth function which is constant 0 near 0 and constant 1 near 1. Let $H'' : M \times I \to T^*M$, where T^*M is the cotangent bundle, be defined by $H''(x,t) = (\omega_1)_x + v(t) \cdot df_x$. Then for every $t \in [0,1]$ $H''_t = H''(\cdot, t)$ is cohomologous to ω_1 with $H''_0 = \omega_1$ and $H''_1 = \omega_2$. Now adapt Milnor [9, §2] to change H'' to $H' : M \times I \to T^*M$ such that H' intersects the zero section transversely. Note that [9, §2] improves a smooth function locally so it carries over to the closed 1-form case and also to a parametrized version. We do not need to change H'' near $M \times \{0,1\}$ so we can assume that $H'_0 = \omega_1$, $H'_1 = \omega_2$ and H'_t is cohomologous to ω_1 for all $t \in [0,1]$.

Now H' intersects the zero section in finitely many intervals and circles. By using a small isotopy of $M \times I$ we can further arrange that the resulting $H : M \times I \to T^*M$ has the following properties:

(1) For all $t \in [0,1]$ H_t is cohomologous to ω_1 and has only finitely many critical points.
(2) Every H_t has at most one degenerate critical point.
(3) Fore only finitely many values of t H_t has a degenerate critical point.

(4) If H_t has a degenerate critical point there is an $\varepsilon_t > 0$ and a $\sigma_t \in \{-1, 1\}$ such that $H_{t+\sigma_t\varepsilon}$ has exactly one more critical point than H_t and $H_{t-\sigma_t\varepsilon}$ has exactly one less critical point for all $0 < \varepsilon < \varepsilon_t$.

So let $t \in (0, 1)$ be such that H_t has a degenerate critical point. Denote this point by $(p, t) \in M \times [0, 1]$. Let U be a neighborhood of (p, t) diffeomorphic to $\mathbb{R}^n \times (t-\varepsilon, t+\varepsilon)$ such that $(t-\varepsilon, t+\varepsilon) \subset [0, 1]$ and $\varepsilon < \varepsilon_t$. Choose an H_t-gradient v_t such that for all nondegenerate critical points the stable and unstable manifolds intersect transversely. By choosing $0 < \delta < \varepsilon$ small enough we can find transverse $H_{t\pm\delta}$-gradients $v_{\pm\delta}$ close to v_t. Without loss of generality we can assume that $H_{t+\delta}$ has two critical points in $\mathbb{R}^n \times \{t + \delta\} \subset U$. Let U_t be the neighborhood of p such that $U_t \times \{t\} = U \cap M \times \{t\}$ and let $h_t : U_t \to \mathbb{R}$ be such that $dh_t = H_t|_{U_t}$. We can change h_t slightly near p to remove the critical point. Hence there is a contractible cylinder neighborhood C of p. But we can also alter h_t slightly near p to introduce the two critical points of $H_{t+\delta}$ and for $\delta > 0$ small enough we stay inside C. Then C has a handlebody decomposition with two critical points. Hence the two critical points have adjacent index and the algebraic coefficient is ± 1.

Let $\{p_1, \ldots, p_l\}$ be the set of nondegenerate critical points of H_t. The closed 1-form H_t pulls back to an exact form dh with $h : \tilde{M} \to \mathbb{R}$. Choose liftings $\tilde{p}_i \in \tilde{M}$ of the critical points. If ω_1 is irrational, we can choose them in $h^{-1}(J)$ where J is an arbitrarily small interval. If ω_1 is rational, we choose them in $h^{-1}([0, r])$, where $r > 0$ is the positive generator of $\operatorname{im} \xi$.

Let $\{p_1^-, \ldots, p_l^-\}$ be the set of critical points of $H_{t-\delta}$ and $\{p_1^+, \ldots, p_l^+, p_{l+1}^+, p_{l+2}^+\}$ be the set of critical points of $H_{t+\delta}$ such that $p_i^\pm$ corresponds to p_i for $i = 1, \ldots, l$. By choosing $\delta > 0$ small enough we can make the $p_i^\pm$ arbitrarily close to p_i. Thus choose the liftings of $p_i^\pm$ close to the $\tilde{p}_i$.

Let us write $C_*(H_{t-\delta}, v_{-\delta}) = C_*^-$ and $C_*(H_{t+\delta}, v_{+\delta}) = C_*^+ \oplus D_*$, where D_* is generated by the two extra critical points. Write the boundary of $C_*(H_{t+\delta}, v_{+\delta})$ in this decomposition as $\begin{pmatrix} \partial^+ & c \\ 0 & \partial^D \end{pmatrix}$. Now look at the matrix of $\psi = \psi_{v_{-\delta}, v_{+\delta}}$. By the definition of ψ using the intersections of stable and unstable manifolds and the fact that the critical points p_i^+ and p_i^- together with their stable and unstable manifolds are close to each other, we get the following description. In the irrational case it looks like $\psi = \begin{pmatrix} I - A \\ \psi_D \end{pmatrix}$ with $A_{ij}(g) = 0$ for $\xi(g) \geq 0$. In the rational case we can assume that $h(\tilde{p}_i) \leq h(\tilde{p}_j)$ for $i \leq j$ and the matrix looks like $\psi = \begin{pmatrix} I - A' - O \\ \psi_D \end{pmatrix}$ with $A'_{ij}(g) = 0$ for $\xi(g) \geq 0$ and O a lower triangular and hence nilpotent matrix. Then $I + O + O^2 + \ldots$ is elementary and $(I - A - O)(I + O + O^2 + \ldots) = I - A$ with A as before. Write $\psi_C = I - A$.

By choosing the liftings of p_{l+1}^+ and p_{l+2}^+ appropriately we get for the coefficient in the boundary of D_* $[\tilde{p}_{l+2}^+ : \tilde{p}_{l+1}^+] = 1 - a$ with $a(g) = 0$ for $\xi(g) \geq 0$.

We now have a short exact sequence of free $\widehat{\mathbb{Z}G}_\xi$ chain complexes

$$0 \longrightarrow C_*^- \xrightarrow{\psi} C_*^+ \oplus D_* \longrightarrow \hat{D}_* \longrightarrow 0$$

where $\hat{D}_*$ is the cokernel of ψ and has rank 2. By Ranicki [20, Prop.1.8] the boundary of $\hat{D}_*$ is $\partial = \partial^D + \psi_D \psi_C^{-1} c$.

We claim that if $\delta > 0$ is small enough, then $\partial = 1 - b$ with $b(g) = 0$ for $\xi(g) \geq 0$. Note that ∂^D is already of that form. Now c counts trajectories from $\tilde{p}^+_{l+2}$ to translates of $\tilde{p}^+_j$ and ψ_D counts broken trajectories from $\tilde{p}^-_j$ to translates of $\tilde{p}^+_{l+1}$. If $\delta > 0$ is small enough $\tilde{p}^+_{l+2}$ and $\tilde{p}^+_{l+1}$ are so close that $(\psi_D \psi_C^{-1} c)(g) \neq 0$ implies $\xi(g) < 0$.

By Ranicki [20, Prop.1.7] we now get that the projection $p : \mathcal{C}(\psi) \to \hat{D}_*$ from the mapping cone of ψ to the cokernel is a chain homotopy equivalence with torsion

$$\tau(p) = \sum_{i=0}^{\infty} (-1)^{i+1} \tau(\psi_C),$$

so $\mathcal{C}(\psi)$ is simple homotopy equivalent to $\hat{D}_*$. But $\hat{D}_*$ has trivial torsion by the claim above. Therefore ψ is a simple chain homotopy equivalence.

Now if there are no degenerate critical points between $H_{t-\delta}$ and $H_{t+\delta}$ the same argument, but easier since $\hat{D}_* = 0$, shows that ψ is a simple isomorphism. By Proposition 4.7 we get that $\psi_{v,w}$ is a simple chain homotopy equivalence. □

Remark 6.3. In the rational case there is a direct proof that φ_v is a simple chain homotopy equivalence using the inverse limit description. See [23, Rm.A.5] for details. That proof is much simpler, but it does not give the irrational case.

That the Novikov complex $C_*(\omega, v)$ is simple chain homotopy equivalent to $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C^\Delta_*(\tilde{M})$ is important for finding a minimal number of critical points for a Morse form within a cohomology class, see Latour [8] or [23]. Other applications of torsion are discussed in the next section.

7. The closed orbit structure of the gradient

The torsions $\tau(\varphi_v)$ and $\tau(\psi_{v,w})$ are already well defined in $K_1(\widehat{\mathbb{Z}G}_\xi)/\langle \tau(\pm g) \,|\, g \in G\rangle$. If we denote by W the subgroup of $K_1(\widehat{\mathbb{Z}G}_\xi)/\langle \tau(\pm g) \,|\, g \in G\rangle$ generated by the trivial units, we get $\tau(\varphi_v), \tau(\psi_{v,w}) \in W$ by Theorem 6.2. In this group the torsion $\tau(\varphi_v)$ does not depend on the triangulation of M, but it does depend on v. In fact it contains information about the closed orbit structure of $-v$. To make this more clear let us define a zeta function of $-v$. Let $\gamma \in H_1(M)$ be a nonzero homology class and let $C_\gamma \subset M \times (0, \infty)$ be the set of closed orbits of $-v$ belonging to γ. By [22, Lm.5.7] and Fuller [4, Th.3] C_γ is an isolated compact set of closed orbits and hence its Fuller index $i(C_\gamma) \in \mathbb{Q}$, see Fuller [4], is defined. Note that $C_\gamma \neq \emptyset$ implies that $\xi(\gamma) < 0$, since $\omega_x(v(x)) < 0$ for $\omega_x \neq 0$. Now $\bar{\eta}(-v) \in \mathbb{Q}^{G_{ab}}$ defined by $\bar{\eta}(-v)(\gamma) = i(C_\gamma)$ satisfies $\bar{\eta}(-v)(\gamma) = 0$ for $\xi(\gamma) \geq 0$ and $\bar{\eta}(-v) \in \widehat{\mathbb{Q}G_{ab}}_\xi$. Then we define the *commutative zeta function* $\bar{\zeta}(-v) = \exp(\bar{\eta}(-v))$, where $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. A priori $\bar{\zeta}(-v)$ is an element of $\widehat{\mathbb{Q}G_{ab}}_\xi$, but one can write down a product formula to see that $\bar{\zeta}(-v) \in \widehat{\mathbb{Z}G_{ab}}_\xi$. But this also follows from the following.

Theorem 7.1. *Let ω be a Morse form on the closed connected smooth manifold M and v a transverse ω-gradient. Let $\overline{M}$ be the universal abelian covering space. Then for the chain homotopy equivalence $\varphi_v : \widehat{\mathbb{Z}G_{ab}}_\xi \otimes_{\mathbb{Z}G_{ab}} C_*(\overline{M}, \omega, v)$ we have* $\det(\tau(\varphi_v)) = \bar{\zeta}(-v)$.

Sketch of proof. Let us assume that $\omega = f^*\alpha$, where $f : M \to S^1$ is a Morse function. As in Section 3 we get the cobordism $(M_N; N, N')$. Pajitnov [13] defines a "cellularity" condition on f-gradients v. Roughly this condition requires a Morse decomposition on N which also gives a decomposition on N' such that points from the i-skeleton of N' either flow into critical points of f with index $\leq i$ or into the i-skeleton of N under the flow of $-\bar{v}$. Here $\bar{v}$ is the lift of v to M_N. A dual condition holds for the flow of $\bar{v}$.

Then the flow of $-\bar{v}$ defines a partially defined function $-\vec{v} : N' \to N$ which behaves cellular. Furthermore the critical points of $\bar{f}|_{M_N}$ together with the decomposition of N gives a decomposition of M_N such that its chain complex calculates $H_*(M_N)$. Also the flow of $-\bar{v}$ defines a cellular map $-\vec{v} : N' \to M_N$ such that on the chain level we get a gradient-like chain approximation $h : C_*(\tilde{N}') \to C_*(\tilde{M}_N)$, see Ranicki [20, §6] for this terminology. Using this chain approximation Ranicki [20, §6] defines a chain homotopy equivalence $p : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \to C_*(\tilde{M}, \omega, v)$ with torsion

$$\tau(p) = \sum_{k=0}^{n-1} (-1)^{k+1} \tau(1 - t h_k^N) \in K_1(\mathbb{Z}H_\theta((t))) = K_1(\widehat{\mathbb{Z}G}_\xi). \tag{4}$$

Here $h_*^N : C_*(\tilde{N}') \to C_*(\tilde{N})$ is determined by the partially defined function $-\vec{v} : N' \to N$. In [21, Prop.4.1] a triangulation of M is constructed such that $p = \varphi_v$. Therefore $\tau(\varphi_v)$ is given in terms of (4) as well. After passing to $\widehat{\mathbb{Z}G_{ab}}_\xi$ we get $\det \tau(\varphi_v) = \prod_{k=0}^{n-1} \det(1 - t h_k^N)^{(-1)^{k+1}}$.

It remains to compare this to $\bar{\zeta}(-v)$. Note that closed orbits of $-v$ correspond to fixed points of iterates of $t^{-1} \circ -\vec{v} : N' \to N'$, where $t^{-1} : \bar{M} \to \bar{M}$ is the covering transformation sending N to N'. It is shown in Pajitnov [14, §8] that the fixed point information contained in $\det \tau(\varphi_v)$ matches the closed orbit information of the zeta function. This proves the theorem for cellular gradients. Now Pajitnov [13] shows such vector fields are C^0-generic. By showing that the torsion and the zeta function depend continuously on v the theorem follows for general v. This is done in [22, §8-§10].

In the case where ω is irrational the theorem can be shown by approximating ω with a rational Morse form, compare [21]. □

Notice that the torsion in (4) is over $\widehat{\mathbb{Z}G}_\xi$, but the zeta function is defined in $\widehat{\mathbb{Z}G_{ab}}_\xi$. Since closed orbits do not give a well defined element of $G = \pi_1(M)$, we cannot define it in $\widehat{\mathbb{Z}G}_\xi$. But the conjugacy class of a closed orbit in G is well defined. Let Γ be the set of conjugacy classes of G and for $\gamma \in \Gamma$ let C_γ be the set of closed orbits of $-v$ belonging to γ. Then $i(C_\gamma) \in \mathbb{Q}$ is well defined and we can define $\eta(-v) \in \widehat{\mathbb{Q}\Gamma}_\xi$ as before. Since $\widehat{\mathbb{Q}\Gamma}_\xi$ is only an abelian group, we cannot take the exponential of $\eta(-v)$.

A different approach to define a noncommutative zeta function was suggested by Geoghegan and Nicas [5] using Hochschild homology. If $\Phi : M \times \mathbb{R} \to M$ is the flow of $-v$, define for every positive integer n a homotopy $F_n : M \times [0, n] \to M$ by restricting Φ. For such an F_n Geoghegan and Nicas [5] define the one-parameter trace $R(F_n) \in HH_1(\mathbb{Z}G)$. Now

$$HH_1(\mathbb{Z}G) \cong \bigoplus_{\gamma \in \Gamma} C(g(\gamma))_{ab}$$

where $g(\gamma) \in G$ represents the conjugacy class γ and $C(g(\gamma))$ is the centralizer of $g(\gamma)$.

A nonzero term in $C(g(\gamma))_{ab}$ detects a closed orbit which represents the conjugacy class γ. The following can be found in [22]. It is possible to complete $HH_1(\mathbb{Z}G)$ to $\widehat{HH}_1(\mathbb{Z}G)_\xi$ such that the sequence $R(F_n)$ converges in $\widehat{HH}_1(\mathbb{Z}G)_\xi$. Then the *noncommutative zeta function* is defined as $\zeta(-v) = \lim_{n\to\infty} R(F_n) \in \widehat{HH}_1(\mathbb{Z}G)_\xi$. Furthermore there is a natural homomorphism $l : \widehat{HH}_1(\mathbb{Z}G)_\xi \to \widehat{\mathbb{R}\Gamma}_\xi$ such that $l(\zeta(-v)) = \eta(-v)$.

To pass from K-theory to Hochschild homology we use the Dennis trace $DT : K_1(\widehat{\mathbb{Z}G_\xi}) \to HH_1(\widehat{\mathbb{Z}G_\xi})$. There is a natural homomorphism $\theta : HH_1(\widehat{\mathbb{Z}G_\xi}) \to \widehat{HH}_1(\mathbb{Z}G)_\xi$. Now $\theta\circ DT$ does not vanish on $\mathrm{im}(\pm G) \subset K_1(\widehat{\mathbb{Z}G_\xi})$, but $\theta\circ DT(\tau(\pm g))$ $= [g] \in C(1_G)_{ab} = G_{ab}$ which is a direct summand of $\widehat{HH}_1(\mathbb{Z}G)_\xi$ which contains no information of $\zeta(-v)$. By projecting this factor away we get a natural homomorphism $\mathfrak{DT} : K_1(\widehat{\mathbb{Z}G_\xi})/\langle\tau(\pm g)\,|\, g \in G\rangle \to \widehat{HH}_1(\mathbb{Z}G)_\xi$ and the generalization of Theorem 7.1 reads

Theorem 7.2. *Let ω be a Morse form on the closed connected smooth manifold M and v a transverse ω-gradient. Then for the chain homotopy equivalence φ_v : $\widehat{\mathbb{Z}G_\xi} \otimes_{\mathbb{Z}G} C_*(\tilde{M}, \omega, v)$ we have $\mathfrak{DT}(\tau(\varphi_v)) = \zeta(-v)$.*

The proof is analogous to the proof of Theorem 7.1, in fact we only need to check that the fixed point information contained in $\mathfrak{DT}(\tau(\varphi_v))$ detects the zeta function. This is similar to the commutative case, but more involved, see [22, §7] for details.

It is worth pointing out that $l \circ \mathfrak{DT}$ has a logarithm property, i.e. for a trivial unit $1 - a$ we have $l \circ \mathfrak{DT}(\tau(1 - a)) = -\sum_{n=1}^{\infty} \frac{\varepsilon(a^n)}{n}$, where $\varepsilon : \widehat{\mathbb{Z}G_\xi} \to \widehat{\mathbb{Z}\Gamma}_\xi$ is augmentation. Denote $\mathfrak{L} = l \circ \mathfrak{DT}$, this homomorphism was obtained by Pajitnov [16] without using Hochschild homology.

Pajitnov [14] defined for cellular gradients of circle valued functions a natural chain homotopy equivalence $\vartheta_v : C(\tilde{M}, f, v) \to \widehat{\mathbb{Z}G_\xi} \otimes_{\mathbb{Z}G} C^\Delta_*(\tilde{M})$. This is done by including the Novikov complex into a complex C'_* which is simply isomorphic to $\widehat{\mathbb{Z}G_\xi} \otimes_{\mathbb{Z}G} C^\Delta_*(\tilde{M})$ as used by Ranicki, compare [14, §7.4]. Composing this with $p : \widehat{\mathbb{Z}G_\xi} \otimes_{\mathbb{Z}G} C^\Delta_*(\tilde{M})$ gives the identity on the Novikov complex. Since both maps are chain homotopy equivalences, they are mutually inverses to each other.

Proposition 7.3. *Let $f : M \to S^1$ be a Morse function and v a transverse cellular f-gradient. Then φ_v and ϑ_v are mutually inverse chain homotopy equivalences.*

Proof. This follows since different smooth triangulations do not change the chain homotopy type of φ_v and for a special triangulation we get $\varphi_v = p$, see the proof of Theorem 7.1. □

Pajitnov [16] obtains

$$\mathfrak{L}(-\tau(\vartheta_v)) = \eta(-v). \tag{5}$$

Of course the proof of Theorem 7.2 uses results from [16], but Proposition 7.3 now shows how (5) follows from Theorem 7.2.

Also Pajitnov [12] defines a chain homotopy equivalence $\vartheta'_v : C_*(\tilde{M}, f, v) \to \widehat{\mathbb{Z}G_\xi}$ $\otimes_{\mathbb{Z}G} C^\Delta_*(\tilde{M})$ without the cellularity assumption. In [17] Pajitnov shows that ϑ'_v and ϑ_v are chain homotopic for cellular v, even though only for the minimal cover $\bar{M}$ instead of the universal cover.

A similar result to Theorem 7.1 has been obtained by Hutchings and Lee [7] and

Hutchings [6]. Both require certain acyclicity conditions on the Novikov complex so the torsion of φ_v is just the difference of two torsions. In particular φ_v is not really needed. Nonetheless [7] contains a version of φ_v, using singular chains instead of simplices. While the method of [7] is similar to the one described in the proof of Theorem 7.1, the method of Hutchings [6] is quite different. There the change in the torsion of the Novikov complex and the zeta function when passing from one Morse form with gradient to another is shown to be equal by using bifurcation analysis. In view of Proposition 4.7 he shows that $\tau(\psi_{v,w})$ detects the difference in zeta functions.

Let us finish by giving an example of a gradient for which the noncommutative zeta function contains more information than the commutative version. For this we need the following theorem which is proven in [23].

Theorem 7.4. *Let G be a finitely presented group, $\xi : G \to \mathbb{Z}$ a homomorphism with finitely presented kernel, $b \in \widehat{\mathbb{Z}G}_\xi$ satisfy $b(g) = 0$ for $\xi(g) \geq 0$ and $n \geq 1$. Then for any closed connected smooth manifold M with $\pi_1(M) = G$ and $\dim M \geq 6$ there is a Morse function $f : M \to S^1$ realizing ξ on fundamental group, a transverse f-gradient v and a $b' \in \widehat{\mathbb{Z}G}_\xi$ with $b'(g) = b(g)$ for $\xi(g) \geq -n$ such that $\tau(\varphi_v) = \tau(1 - b') \in K_1(\widehat{\mathbb{Z}G}_\xi)/\langle \tau(\pm g) \,|\, g \in G\rangle$.*

Now let N be a nontrivial finitely presented nonabelian group. Then the projection $\xi : G = N \times \mathbb{Z} \to \mathbb{Z}$ satisfies the conditions of Theorem 7.4. Let $n \in [N, N] - \{1_N\}$. Then $g = (n, -1)$ and $h = (1_N, -1)$ are not conjugated in G, but project to the same element in $G_{ab} = H_{ab} \times \mathbb{Z}$. Look at $\tau = \tau((1 - g)(1 - h)^{-1}) \in W$. Then $\varepsilon(\tau) = 0 \in K_1(\mathbb{Z}H_{ab}((t)))$. To see that $\mathfrak{D}\mathfrak{T}(\tau) \neq 0$ we have the next Lemma.

Lemma 7.5. *Let G be a group and $\xi : G \to \mathbb{R}$ a homomorphism. For $g \in G$ let $p_g : \widehat{HH}_1(\mathbb{Z}G)_\xi \to C(g)_{ab}$ be the projection. If $\xi(g) < 0$, then $p_g \circ \mathfrak{D}\mathfrak{T}(\tau(1-g)) \neq 0$ and $p_h \circ \mathfrak{D}\mathfrak{T}(\tau(1 - g)) = 0$ if h is not conjugated to g^n for any positive n.*

Proof. By [22, §4] $\mathfrak{D}\mathfrak{T}(\tau(1 - g))$ is represented by a 1-chain $-\sum_{k=0}^{\infty} g^k \otimes g$ and $p_g([g^k \otimes g]) = [g] \in C(g)_{ab}$ for $k = 0$ and 0 for $k > 0$. Also $p_h([g^k \otimes g]) = 0$ if h is not conjugated to g^{k+1}. To see that $[g] \neq 0 \in C(g)_{ab}$ note that $\xi : G \to \mathbb{R}$ restricts to $\xi : C(g) \to \mathbb{R}$ which induces $\bar{\xi} : C(g)_{ab} \to \mathbb{R}$ and clearly $\bar{\xi}([g]) = \xi(g) < 0$ by assumption. □

In our situation we get $p_g(\mathfrak{D}\mathfrak{T}(\tau)) = [g]$ and $p_h(\mathfrak{D}\mathfrak{T}(\tau)) = -[h]$. Now apply Theorem 7.4 to get a manifold M, a Morse function $f : M \to S^1$ and a transverse f-gradient v such that $\tau(\varphi_v) = \tau - \tau(1 - b)$ where $b(g) = 0$ for $\xi(g) \geq -1$. Then $\zeta(-v)$ detects two closed orbits corresponding to g and h while $\bar{\zeta}(-v)$ does not detect any closed orbits corresponding to $-1 \in \mathbb{Z} = H_1(M)$.

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