OBSTRUCTIONS TO HOMOTOPY INVARIANCE IN PARAMETRIZED FIXED POINT THEORY

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Abstract. In "zero-parameter" or classical Nielsen fixed point theory one studies $\mathrm{Fix}(f) := \{x \in X \mid f(x) = x\}$ where $f : X \to X$ is a map. In case X is an oriented compact manifold and f is transverse to the identity map, id_X, $\mathrm{Fix}(f)$ is a finite set each of whose elements carries a natural sign, ± 1, the *index* of that fixed point. The set $\mathrm{Fix}(f)$ is partitioned into *Nielsen classes*. Adding the indices within a given Nielsen class yields the *fixed point index* of that class. The number of classes with nonzero fixed point index is called the *Nielsen number* of f. One-parameter fixed point theory is the study of the fixed point set, $\mathrm{Fix}(F) := \{(x,t) \in X \times I \mid F(x,t) = x\}$, of a homotopy $F : X \times I \to X$. In the case where X is an oriented compact manifold and F is transverse to the projection map $p : X \times I \to X$, $\mathrm{Fix}(X)$ consists of naturally oriented circles in the interior of $X \times I$ and naturally oriented arcs whose endpoints lie in $X \times \{0,1\}$. We give a summary of one-parameter fixed point theory, in analogy to the classical theory, as developed by the first two authors. While classical fixed point theory is a "homotopy invariant" theory, homotopy invariance in the one-parameter theory is obstructed by "torsion" type invariants. We discuss this phenomenon and give some new examples.

Introduction

One-parameter fixed point theory is the study of the fixed point set, $\mathrm{Fix}(F) := \{(x,t) \in X \times I \mid F(x,t) = x\}$, of a homotopy $F : X \times I \to X$. In the case where X is an oriented compact manifold and F is transverse to the projection map $p : X \times I \to X$, $\mathrm{Fix}(X)$ consists of naturally oriented circles in the interior of $X \times I$ and naturally oriented arcs whose endpoints lie in $X \times \{0,1\}$ as shown schematically in Figure 1.

Before organizing the information suggested by Figure 1, we motivate our discussion by recalling "0-parameter" or classical Nielsen fixed point theory. There one studies $\mathrm{Fix}(f) := \{x \in X \mid f(x) = x\}$ where $f : X \to X$ is a map. In case X is an oriented compact manifold and f is transverse to the identity map, id_X, $\mathrm{Fix}(f)$ is a finite set each of whose elements carries a natural sign, ± 1, the *index* of that fixed point. The sum of the indices is the Lefschetz number, $L(f)$. A certain equivalence relation (see §1) on the set $\mathrm{Fix}(f)$ partitions it into *Nielsen classes*. Adding the indices within a given Nielsen class yields the *fixed point index* of that class. The number of classes with nonzero fixed point index is denoted by $N(f)$ and is called the *Nielsen number*

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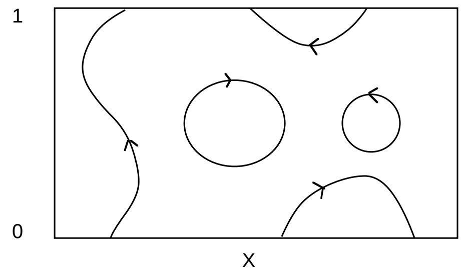
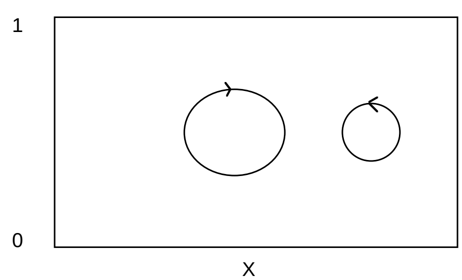


of f. Thus $L(f)$, an invariant of the homotopy class of f, decomposes as a sum of $N(f)$ integers, each of which is also an invariant of the homotopy class of f. The non-negative integer $N(f)$ is a lower bound for the number of fixed points of maps homotopic to f. If the information "Nielsen classes and their fixed point indices" is organized in the correct algebraic setting then it is also homotopy invariant in another sense: if X' is another oriented compact manifold and $h : X \to X'$ is a homotopy equivalence then the information for $f : X \to X$ is the same as the information for $h \circ f \circ k : X' \to X'$ where k is a homotopy inverse for h. Summarizing, we say *Nielsen fixed point theory is a homotopy invariant theory.*

The geometric theory just outlined has an algebraic formulation in which the space X is a finite oriented CW complex, $f : X \to X$ is a cellular map, and $L(f) = \text{trace}(f_*)$ where $f_* : C_*(X) \to C_*(X)$ is the morphism induced on cellular chains (alternating signs built in). For the more refined version involving Nielsen classes, let $G := \pi_1(X, v)$ and let f and a choice of basepath from v to $f(v)$ induce $\phi : G \to G$. Using cellular chains in the universal cover of X, calculate $\text{trace}(\tilde{f}_*) \in \mathbb{Z}G$. The equivalence relation of "semiconjugacy" on G ($g \sim hg\phi(h^{-1})$) defines a set, G_ϕ, of equivalence classes and the image, $R(f)$, of $\text{trace}(\tilde{f}_*)$ in $\mathbb{Z}G_\phi$ is the *Reidemeister trace* of f. This $R(f)$ contains the same information as the Nielsen classes and their fixed point indices. [1]

We return to the one-parameter situation of Figure 1. To keep the geometric discussion in this Introduction simple we will only discuss the case where there are no arcs in Figure 1. This is illustrated in Figure 2. (The general case is dealt with in subsequent sections.)



A complete theory analogous to the classical theory outlined above was presented in [GN$_1$]. In the geometric theory, the role of $L(f)$ is played by $L(F) \in H_1(X) \cong G_{\text{ab}}$,

[1]The details of Nielsen fixed point theory can be found in [Br] and [Ki]. Summaries compatible with the viewpoint of the present paper are found in §1(B) of [GN$_1$] and [Ge].

where $G_{\text{ab}} := G/[G : G]$ is the abelianization of G. This is the projection to X of the 1–dimensional homology class in $X \times I$ indicated in Figure 2. To obtain a geometric analog of $R(f)$ one partitions the circles in Figure 2 into Nielsen classes as in the classical case (see §1(B)), and one regards the circles in a single class as 1–cycles in a suitable covering space of X, a different covering space for each class. So $R(F)$ lies in $\bigoplus_\alpha H_1(\tilde{X}_\alpha)$ where the $\tilde{X}_\alpha$'s are covering spaces of X associated to the classes.

For our purposes here, this geometric theory is only mentioned to explain intuitively what is going on. This paper is about the equivalent algebraic theory. We have a cellular homotopy $F : X \times I \to X$ where X is a finite oriented CW complex. The algebraic version of $R(F)$ lies in $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(\partial F))$. Here, $(\mathbb{Z}G)^\phi$ is the $\mathbb{Z}G$–$\mathbb{Z}G$ bimodule with left action $h \cdot g := hg$ and right action $g \cdot h := g\phi(h)$; $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ denotes the Hochschild homology of $\mathbb{Z}G$ with coefficients in the bimodule $(\mathbb{Z}G)^\phi$, and the "relativization" $G_\phi(\partial F)$ only occurs in the presence of arcs in Figure 1, i.e., when $\text{Fix}(F_0)$ or $\text{Fix}(F_1)$ is non-empty; so this would be ordinary Hochschild homology in the situation of Figure 2 (for details see Definition 1.8). This paper concerns the invariance properties of $R(F)$. If F is homotopic to F' rel $X \times \{0, 1\}$ then it is routine to show that $R(F) = R(F')$ (see Corollary 4.3 of [GN_1]). The interesting question is: if X' is another finite oriented CW complex and if $h : X \to X'$ is some sort of equivalence and if the following diagram commutes up to homotopy rel $X \times \{0, 1\}$

$$\begin{tikzcd} X \times I \arrow[r, "F"] \arrow[d, "h\times\text{id}"'] & X \arrow[d, "h"] \\ X' \times I \arrow[r, "F'"] & X' \end{tikzcd}$$

is it true that $h_\dagger(R(F)) = R(F')$ where the homomorphism $h_\dagger$ between Hochschild homology groups is induced by h (see §2)? The answer depends on what sort of equivalence h is:

- If h is a homotopy equivalence, the general answer is NO. $R(F')$ is not a homotopy invariant. This phenomenon was discussed in §7 of [GN_1] (also see Remark 5.8 of [GN_1]) and is briefly recalled here in Example 2.9.
- If h is a simple homotopy equivalence, the general answer is still NO: $R(F)$ is not a simple homotopy invariant. A counterexample is given in §3. However, simple homotopy invariance holds in a special case: if F and F' are self homotopies of the identity then $h_\dagger(R(F)) = R(F')$; see Theorem 2.8.
- If h is the identity map from the CW complex X to a subdivision X' of X the answer is YES: We prove $R(F)$ is a subdivision (i.e., combinatorial) invariant (Theorem 4.6). Since the identity map $X \to X'$ is a simple homotopy equivalence, this is another special case where simple homotopy invariance is valid.
- If h is a homeomorphism, the answer is unknown.

We note that in Theorem 4.5 of [GN_1] it was incorrectly asserted that $R(F)$ is a simple homotopy invariant in general; see §3. Lemma 4.8 of [GN_1] on which the proof of Theorem 4.5 of [GN_1] depends is not valid in the generality claimed there. Simple homotopy invariance of $R(F)$ in the general case was never used in [GN_1]

nor in the subsequent papers [2] [GN_2] and [GN_3] which build on [GN_1]. However, subdivision invariance, as in (iii) above, was used, and was presented in [GN_1] as a corollary of simple homotopy invariance. It now requires a direct proof, which is given here in Theorem 4.6.

The case of a homotopy $F : X \times I \to X$ in which $F_0 = F_1 = \mathrm{id}_X$, where simple homotopy invariance does hold, is important; it is analogous to the classical fixed point theory of $\mathrm{id}_X : X \to X$, i.e., the theory of the Euler characteristic, $\chi(X)$. In this case, $R(F)$ becomes a "higher Euler characteristic", a new invariant with interesting applications to circle actions on manifolds. This is discussed at length in [GN_2]. A proof of simple homotopy invariance in this context, independent of [GN_1], is given in Theorem 2.10 of [GN_2]; furthermore, we note that the original proof in [GN_1] is valid in this restricted situation.

What should replace simple homotopy invariance in the general case? Theorem 2.6 shows that the obstruction to homotopy invariance depends only on F_0 and F_1 (and their associated basepaths). The results of §2 suggest there may exist an appropriate notion of an "f-twisted simple homotopy equivalence" but this is an open problem.

This paper is an addendum to (and a correction of) [GN_1]. However we have tried to make it readable in itself even though [GN_1] is a reference for some of the proofs. For the convenience of the reader we repeat some basic definitions in §1. We concentrate here on the algebraic side of the theory; for a treatment of the broader subject including geometric aspects, see §1 of [GN_1].

Two of us (R.G. and A.N.) would like to thank our coauthor Dirk Schütz whose careful reading uncovered the error in Lemma 4.8 of [GN_1], and who immediately went on to supply a proof of combinatorial invariance, given here in a suitably adapted form in §4.

§1. The One-Parameter Trace of a Homotopy

In this section we review the definition of the 1–parameter trace of a cellular homotopy $F : X \times I \to X$ as introduced in [GN_1]. We begin with some background material concerning Hochschild homology and traces.

(A) Algebraic Preliminaries.

Let R be a commutative ground ring and let S be an associative R–algebra with unit. If M is an S–S bimodule (i.e., a left and right S–module satisfying $(s_1m)s_2 = s_1(ms_2)$ for all $m \in M$, and $s_1, s_2 \in S$), the *Hochschild chain complex* $(C_*(S, M), d)$ consists of $C_n(S, M) = S^{\otimes n} \otimes M$ where $S^{\otimes n}$ is the tensor product of n copies of S and

$$\begin{aligned} d(s_1 \otimes \cdots \otimes s_n \otimes m) \quad = \quad & s_2 \otimes \cdots \otimes s_n \otimes ms_1 \\ & + \sum_{i=1}^{n-1} (-1)^i s_1 \otimes \cdots \otimes s_is_{i+1} \otimes \cdots \otimes s_n \otimes m \\ & + (-1)^n s_1 \otimes \cdots \otimes s_{n-1} \otimes s_nm. \end{aligned}$$

[2] While the assertion of simple homotopy invariance of $R(F)$ is repeated in Theorem 1.12 of [GN_3] only subdivision invariance is used in that paper.

The tensor products are taken over R. The n–th homology of this complex is the n–th *Hochschild homology of S with coefficient bimodule M*. It is denoted by $HH_n(S, M)$. If $M = S$ with the standard S–S bimodule structure then it is customary to write $HH_n(S)$ for $HH_n(S, M)$.

We will be concerned mainly with HH_1 which is computed from

$$\begin{array}{ccccccc} \cdots \longrightarrow & S \otimes S \otimes M & \xrightarrow{d} & S \otimes M & \xrightarrow{d} & M & \\ & s_1 \otimes s_2 \otimes m & \mapsto & s_2 \otimes ms_1 - s_1s_2 \otimes m + s_1 \otimes s_2m & & & \\ & & & s \otimes m & \mapsto & ms - sm & \end{array}$$

The following observation will be useful:

Lemma 1.1. *Let $1 \in S$ be the unit element and let $m \in M$. Then the 1–chain $1 \otimes m$ is a boundary.*

Proof. $d(1 \otimes 1 \otimes m) = 1 \otimes m - 1 \otimes m + 1 \otimes m = 1 \otimes m$. □

If A is a $m \times n$ matrix over S and B is a $n \times m$ matrix over M, we define $A \otimes B$ to be the $m \times m$ matrix with entries in the R–module $S \otimes M$ given by

$$(A \otimes B)_{ij} = \sum_{k=1}^{n} A_{ik} \otimes B_{kj}.$$

The *trace* of $A \otimes B$, written $\operatorname{trace}(A \otimes B)$, is $\sum_{i=1}^{m} \sum_{k=1}^{n} A_{ik} \otimes B_{ki}$ which we interpret as a Hochschild 1–chain. Observe that the 1–chain $\operatorname{trace}(A \otimes B)$ is a cycle if and only if $\operatorname{trace}(AB) = \operatorname{trace}(BA)$, in which case we denote its homology class by $T_1(A \otimes B) \in HH_1(S, M)$.

We use Hochschild homology in the following situation. Let G be a group and $\phi : G \to G$ an endomorphism. Also denote by ϕ the induced ring homomorphism $\mathbb{Z}G \to \mathbb{Z}G$. Take the ground ring R to be $\mathbb{Z}$, the ring S to be $\mathbb{Z}G$, and the bimodule M to be the $\mathbb{Z}G$–$\mathbb{Z}G$ bimodule whose underlying abelian group is $\mathbb{Z}G$ and whose bimodule structure is given by $g \cdot m := gm$ and $m \cdot g := m\phi(g)$. We denote this bimodule by $(\mathbb{Z}G)^\phi$.

Elements g_1 and g_2 of G are *semiconjugate* if and only if there exists $g \in G$ such that $g_1 = gg_2\phi(g^{-1})$. We write $C(g)$ for the semiconjugacy class containing g and G_ϕ for the set of semiconjugacy classes. The partition of G into the union of its semiconjugacy classes induces a direct sum decomposition of $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ as follows: each generating chain $c = g_1 \otimes \cdots \otimes g_n \otimes m$ can be written in *canonical form* as $g_1 \otimes \cdots \otimes g_n \otimes g_n^{-1} \cdots g_1^{-1} g$ and we call $g := g_1 \cdots g_n m \in G$ the *marker* of c. All the generating chains occurring in the boundary $d(c)$ are easily seen to have markers in the same semiconjugacy class $C(g)$ when put into canonical form. For $C \in G_\phi$ let $C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ be the subgroup of $C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ generated by those generating chains whose markers lie in C. The decomposition $(\mathbb{Z}G)^\phi \cong \bigoplus_{C \in G_\phi} \mathbb{Z}C$ as a direct sum of abelian groups determines a decomposition of chain complexes $C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$. There results a natural isomorphism $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ where the summand $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ corresponds to the homology classes of Hochschild cycles marked by the elements of C. We call this summand the *C–component*.

Let $\overline{(\mathbb{Z}G)^\phi}$ be the left G–module whose underlying abelian group is $\mathbb{Z}G$ and whose left module structure is given by $g \cdot m := gm\phi(g^{-1})$. There is a natural isomorphism $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong H_*(G, \overline{(\mathbb{Z}G)^\phi})$, which is induced from an isomorphism of the Hochschild complex to the bar complex for computing group homology, see Theorem 1.d of [I]. The decomposition $\overline{(\mathbb{Z}G)^\phi} \cong \bigoplus_{C \in G_\phi} \mathbb{Z}C$ is a direct sum of left $\mathbb{Z}G$ modules, inducing a direct sum decomposition $H_*(G, \overline{(\mathbb{Z}G)^\phi}) \cong \bigoplus_{C \in G_\phi} H_*(G, \mathbb{Z}C)$. Choosing representatives $g_C \in C$ we have an isomorphism of left $\mathbb{Z}G$ modules $\mathbb{Z}C \cong \mathbb{Z}(G/Z(g_C))$ where $Z(h) = \{g \in G \mid h = gh\phi(g^{-1})\}$ denotes the *semicentralizer* of $h \in G$. Since $H_*(G, \mathbb{Z}(G/Z(g_C)))$ is naturally isomorphic to $H_*(Z(g_C))$, we obtain:

Proposition 1.2. *There is a natural isomorphism*

$$HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} H_*(Z(g_C))$$

and $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ corresponds to the summand $H_*(Z(g_C))$ under this identification. □

We now discuss the "relativization" mentioned in the Introduction. It may happen in topological applications that $\mathrm{trace}(A \otimes B) \in C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ is not a cycle, but that for an appropriate geometrically defined $J \subset G_\phi$ its C–component $[\mathrm{trace}(A \otimes B)]_C \in C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ is a cycle for all $C \notin J$. We define

$$HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi; J) := \bigoplus_{C \in G_\phi - J} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$$

regarded as a subgroup of $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ and we write $T_1(A \otimes B; J)$ for the element of $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; J)$ whose C–component is represented by $[\mathrm{trace}(A \otimes B)]_C$ for each $C \in G_\phi - J$. If $J \subset J' \subset G_\phi$ then the projection

$$\bigoplus_{C \in G_\phi - J} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \longrightarrow \bigoplus_{C \in G_\phi - J'} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$$

defines a homomorphism $p : HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi; J) \to HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi; J')$.

Next, we discuss morphisms between Hochschild groups. Suppose $\alpha : G \to H$, $\phi : G \to G$, and $\psi : H \to H$ are group homomorphisms such that $\alpha\phi = \psi\alpha$. Such an α induces a chain map $\alpha_* : C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \to C_*(\mathbb{Z}H, (\mathbb{Z}H)^\psi)$ given by $\alpha_*(s_1 \otimes \cdots \otimes s_n \otimes m) = \alpha(s_1) \otimes \cdots \otimes \alpha(s_n) \otimes \alpha(m)$. Note that α induces a function, also denoted α, $G_\phi \xrightarrow{\alpha} H_\psi$ given by $\alpha(C(g)) = C(\alpha(g))$. If α is an isomorphism of groups and $J \subset G_\phi$ then $\alpha(G_\phi - J) = H_\psi - \alpha(J)$ and there is an isomorphism

$$\alpha_* : HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi; J) \quad \xrightarrow{\cong} \quad HH_*(\mathbb{Z}H, (\mathbb{Z}H)^\psi; \alpha(J)).$$

There is a left action of the center of G, $\mathrm{Center}(G)$, on $HH_*(\mathbb{Z}G)$ which we will use in §2. At the level of chains it is defined by

$$\omega \cdot \big(g_1 \otimes \cdots \otimes g_n \otimes m\big) \;=\; g_1 \otimes \cdots \otimes g_n \otimes (m\omega^{-1}) \tag{1.3}$$

where $\omega \in \mathrm{Center}(G)$. One easily checks that this action is compatible with d and hence makes $HH_*(\mathbb{Z}G)$ into a left $\mathrm{Center}(G)$–module. The summand $HH_*(\mathbb{Z}G)_C$ is taken by the left action of ω isomorphically onto the summand $HH_*(\mathbb{Z}G)_{C\omega^{-1}}$ where $C\omega^{-1}$ is the conjugacy class $\{g\omega^{-1} \mid g \in C\}$.

We recall the definition of K_1 of a ring. Let $GL(n,R)$ denote the general linear group consisting of all $n \times n$ invertible matrices over R, and let $GL(R)$ be the direct limit of the sequence $GL(1,R) \subset GL(2,R) \subset \cdots$. Then $K_1(R) := H_1(GL(R))$.

Definition 1.4. The *Dennis trace* homomorphism $\mathrm{DT} : K_1(R) \to HH_1(R)$ is defined as follows. If $\alpha \in K_1(R)$ is represented by an invertible $n \times n$ matrix A then $\mathrm{DT}(\alpha) = T_1(A \otimes A^{-1})$ (see Chapter 1 of [I]).

In case $R = \mathbb{Z}G$, let $\pm G \subset GL(1,\mathbb{Z}G)$ be the subgroup consisting of 1×1 matrices of the form $[\pm g]$, $g \in G$. The cokernel of the natural homomorphism $\pm G \to K_1(\mathbb{Z}G)$ is called the *Whitehead group of* G and is denoted by $\mathrm{Wh}(G)$.

Proposition 1.5. *The image of the composite homomorphism:*

$$\pm G \xrightarrow{i} K_1(\mathbb{Z}G) \xrightarrow{\mathrm{DT}} HH_1(\mathbb{Z}G)$$

lies in $HH_1(\mathbb{Z}G)_{C(1)}$.

Proof. For $g \in G$, $\mathrm{DT}(i(\pm g)) =$ homology class of $g \otimes g^{-1} \in HH_1(\mathbb{Z}G)_{C(1)}$. □

Proposition 1.5 implies that the Dennis trace induces a homomorphism

$$\mathrm{DT} : \mathrm{Wh}(G) \longrightarrow HH_1(\mathbb{Z}G; \{C(1)\}) \tag{1.6}$$

where we write $HH_1(\mathbb{Z}G; J)$ for $HH_1(\mathbb{Z}G; (\mathbb{Z}G)^{\mathrm{id}}; J)$ with $J \subset G_{\mathrm{id}}$.

We will also use an analog of the Dennis trace in §2. Let $M(n,\mathbb{Z}G)$ denote the abelian group consisting of all $n \times n$ matrices over $\mathbb{Z}G$, and let $M(\mathbb{Z}G)$ be the direct limit of the sequence $M(1,\mathbb{Z}G) \subset M(2,\mathbb{Z}G) \subset \cdots$. Given an endomorphism $\phi : G \to G$, the group $GL(\mathbb{Z}G)$ acts on $M(\mathbb{Z}G)$ on the left by $m.u = um\phi(u^{-1})$ where $m \in M(\mathbb{Z}G)$, $u \in GL(\mathbb{Z}G)$ and $um\phi(u^{-1})$ is the matrix product of $\mathbb{Z}G$-matrices. Let $M(\mathbb{Z}G)^{\phi}$ denote $M(\mathbb{Z}G)$ with this left $GL(\mathbb{Z}G)$ module structure. Define a homomorphism

$$\mathcal{D} : H_1(GL(\mathbb{Z}G); M(\mathbb{Z}G)^{\phi}) \longrightarrow HH_1(\mathbb{Z}G; (\mathbb{Z}G)^{\phi}) \tag{1.7}$$

by $\mathcal{D}(\alpha) = \sum_i \mathrm{trace}(u_i \otimes u_i^{-1} m_i)$ where $\alpha \in H_1(GL(\mathbb{Z}G); M(\mathbb{Z}G)^{\phi})$ is represented by the 1–cycle $\sum_i (u_i - 1) \otimes m_i$ for some $u_i \in GL(n,\mathbb{Z}G) \subset GL(\mathbb{Z}G)$ and $m_i \in M(n,\mathbb{Z}G) \subset M(\mathbb{Z}G)$ (recall that the first homology of a group K with coefficients in a left K-module B is naturally isomorphic to $\ker(IK \otimes_{\mathbb{Z}K} B \to B)$, $(x-1)\otimes b \mapsto xb - b$, $x \in K, b \in B$, where IK is the augmentation ideal of $\mathbb{Z}K$).

(B) The One-Parameter Trace of a Homotopy.

Let X be a finite connected oriented CW complex and let $I = [0,1]$ be endowed with its usual CW structure and orientation of cells. Let $F : X \times I \to X$ be a cellular homotopy, where $X \times I$ has the product CW structure and its cells are given the product orientation.

Choose a basepoint $(v,0) \in X \times I$ and choose a basepath τ from v to $F(v,0)$. We identify $\pi_1(X \times I, (v,0))$ with $G := \pi_1(X,v)$ via the isomorphism induced by the projection $p : X \times I \to X$. In particular, we write $\phi : G \to G$ for the homomorphism

$$\pi_1(X \times I, (v,0)) \xrightarrow{F_{\#}} \pi_1(X, F(v,0)) \xrightarrow{(\tau^{-1})} \pi_1(X, v).$$

Let $\tilde{\tau}$ be the lift of the basepath τ which starts at the basepoint, $\tilde{v} \in \tilde{X}$, and let $\tilde{F}$ be the unique lift of F mapping $(\tilde{v}, 0)$ to $\tilde{\tau}(1)$. $\tilde{F}$ induces a chain homotopy $\tilde{D}_k : C_k(\tilde{X}) \to C_{k+1}(\tilde{X})$ given as follows:

Sign Convention. If $\tilde{e}$ is an oriented k–cell of $\tilde{X}$ then $\tilde{D}_k(\tilde{e})$ is the $(k+1)$–chain $(-1)^{k+1}\tilde{F}_*(\tilde{e} \times I) \in C_{k+1}(\tilde{X})$, where $\tilde{e} \times I$ is given the product orientation. This is consistent with the convention that if $E_{i,\epsilon}$ is the face of the cube $I^n = [0,1]^n$ obtained by holding the i^{th} coordinate fixed at $\epsilon = 0$ or 1, then the incidence number $[I^n : E_{i,\epsilon}]$ is $(-1)^{i+\epsilon}$. At the level of cellular n–chains, we have $\partial_n I^n = \sum_{i,\epsilon}[I^n : E_{i,\epsilon}]E_{i,\epsilon}$.

This satisfies $\tilde{D}_k(\tilde{e}g) = \tilde{D}_k(\tilde{e})\phi(g)$. The boundary $\tilde{\partial}_k : C_k(\tilde{X}) \to C_{k-1}(\tilde{X})$, however, satisfies $\tilde{\partial}_k(\tilde{e}g) = \tilde{\partial}_k(\tilde{e})g$. Define endomorphisms of $\oplus_k C_k(\tilde{X})$ by $\tilde{D}_* = \oplus_k(-1)^{k+1}\tilde{D}_k$, $\tilde{\partial}_* = \oplus_k \tilde{\partial}_k$, $(\tilde{F}_0)_* = \oplus_k(-1)^k(\tilde{F}_0)_k$, and $(\tilde{F}_1)_* = \oplus_k(-1)^k(\tilde{F}_1)_k$. We reuse the symbols $\tilde{D}_*$, $\tilde{\partial}_*$, $(\tilde{F}_0)_*$, and $(\tilde{F}_1)_*$ for the matrices of the corresponding endomorphisms. The chain homotopy relation yields the matrix equation:

$$\tilde{D}_*\phi(\tilde{\partial}_*) - \tilde{\partial}_*\tilde{D}_* = (\tilde{F}_0)_* - (\tilde{F}_1)_*.$$

The minus sign appearing on the left arises from our convention concerning the alternation of signs. Note that the entry of the matrix $\tilde{D}$ corresponding to an n–cell $\tilde{e}_1$ and an $(n+1)$–cell $\tilde{e}_2$ is the coefficient of $\tilde{e}_2$ in the $(n+1)$–chain $\tilde{F}_*(\tilde{e}_1 \times I)$.

Let ω be the path given by $\omega(t) = F(v,t)$. Then $\tau\omega$ is a path from v to $F_1(v)$. Note that it is this path which must be used to determine the lift $\tilde{F}_1$ of F_1 and so even if $F_0 = F_1$ it is possible that $\tilde{F}_0 \neq \tilde{F}_1$.

In the language of §1(A), consider trace$(\tilde{\partial}_* \otimes \tilde{D}_*) \in \mathbb{Z}G \otimes (\mathbb{Z}G)^\phi$. This is a Hochschild 1–chain whose boundary is

$$\operatorname{trace}(\tilde{D}_*\phi(\tilde{\partial}_*) - \tilde{\partial}_*\tilde{D}_*) = \operatorname{trace}((\tilde{F}_0)_* - (\tilde{F}_1)_*).$$

The latter might not be zero, so trace$(\tilde{\partial}_* \otimes \tilde{D}_*)$ might not be a cycle; but in the important special case in which F_0 and F_1 have no fixed points then trace$(\tilde{\partial}_* \otimes \tilde{D}_*)$ is a cycle because by Theorem 2.6 of [GN$_1$] trace$((\tilde{F}_0)_*) =$ trace$((\tilde{F}_1)_*) = 0$ and in this situation the invariant of interest to us is $T_1(\tilde{\partial}_* \otimes \tilde{D}_*) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$.

For the general case, recall that we are studying how Fix(F) is altered by homotopies of F rel $X \times \{0,1\}$. Since this is a relative problem it makes sense to remove the influence of F_0 and F_1. First, we recall how semiconjugacy classes are associated to the fixed points of a map $f : X \to X$. Let η be a basepath from v to $f(v)$. Two fixed points x and y of f are *in the same fixed point class* if and only if for some path ν from x to y, the loop $\nu(f \circ \nu)^{-1}$ is homotopically trivial. This defines an equivalence relation on the set of fixed points, Fix(f). There is an injective function Φ from the set of fixed point classes of f into G_ϕ: the class containing x is mapped to the semiconjugacy class containing $[\mu(f \circ \mu)^{-1}\eta^{-1}]$, where μ is any path from the basepoint v to x. We say x is *associated with* that semiconjugacy class. We write $G_\phi(f)$ for the image of Φ in G_ϕ. It is straightforward to check that Φ is well-defined, that Fix(f) is thus partitioned into only finitely many fixed point classes, and that fixed points in the same path component of Fix(f) are in the same fixed point class. Let $G_\phi(\partial F) := G_\phi(F_0) \cup G_\phi(F_1)$, i.e., the subset of G_ϕ consisting of semiconjugacy classes associated to fixed points of F_0 or F_1 (where we are using the basepath τ for F_0 and the basepath $\tau\omega$ for F_1).

Definition 1.8. The one-parameter trace of F is

$$R(F,\tau) := T_1(\tilde{\partial}_* \otimes \tilde{D}_*; G_\phi(\partial F)) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(\partial F)).$$

§2. Obstructions to Homotopy Invariance and Proof of Simple Homotopy Invariance in a Special Case

Let (Y,X) be a finite CW pair where Y is connected and the inclusion $i : X \hookrightarrow Y$ is a homotopy equivalence. Choose a basepoint $v \in X$ and let $G := \pi_1(Y,v)$. Let $f : (Y,X) \to (Y,X)$ be a cellular map of pairs. Choose a basepath τ in X from v to $f(v)$. Let $\phi : G \to G$ be the homomorphism induced by f and τ. Recall that $G_\phi(f)$ denotes the set of those ϕ-semiconjugacy classes of G which are associated with fixed points of f. Let $H : Y \times I \to Y$ be a strong deformation retraction of Y into X, i.e., $H_0 = \mathrm{id}$, $H_1 = ir$ where $r : Y \to X$ is a retraction, and $H(x,t) = x$ for all $x \in X$. Consider the homotopy of pairs $U := H \circ (f \times \mathrm{id}) : (Y,X) \times I \to (Y,X)$. Since $U_0 = f$ and $U_1|_X = f|_X$, $\mathrm{Fix}(U_1) = \mathrm{Fix}(f) \cap X$. Hence $G_\phi(\partial U) = G_\phi(f)$.

Definition 2.1. $\mathcal{E}(f,\tau) := R(U,\tau) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f))$.

Proposition 2.2. *$\mathcal{E}(f,\tau)$ is independent of the choice of the strong deformation retraction H.*

Proof. Let $\tilde{Y}$ be the universal cover of Y and $\tilde{X} \subset \tilde{Y}$ the universal cover of X. Let $(C_*(\tilde{Y}), \tilde{\partial}'_*)$ and $(C_*(\tilde{X}), \tilde{\partial}_*)$ be the corresponding cellular chain complexes. The relative chain complex of the pair $(\tilde{Y},\tilde{X})$, $(C_*(\tilde{Y},\tilde{X}), \tilde{\partial}''_*)$ is contractible. We have a strictly commutative diagram of $\mathbb{Z}G$–complexes:

$$\begin{tikzcd}
C_*(\tilde{X}) \arrow[r] \arrow[d, "\tilde{D}_*"'] & C_*(\tilde{Y}) \arrow[r] \arrow[d, "\tilde{D}'_*"'] & C_*(\tilde{Y},\tilde{X}) \arrow[d, "\tilde{D}''_*"'] \\
C_*(\tilde{X}) \arrow[r] & C_*(\tilde{Y}) \arrow[r] & C_*(\tilde{Y},\tilde{X})
\end{tikzcd}$$

where $\tilde{D}'_*$ is the chain homotopy associated to $U := H \circ (f \times \mathrm{id})$, $\tilde{D}_*$ is the chain homotopy associated to $U|_{X\times I}$, and $\tilde{D}''_*$ is the induced chain homotopy on the relative chain complex. Note that $\tilde{D}_* = 0$ since $U|_{X\times I}$ is a constant homotopy. By Proposition 3.5 and Addendum 3.6 of [GN$_1$],

$$\mathrm{trace}(\tilde{\partial}'_* \otimes \tilde{D}'_*) - \mathrm{trace}(\tilde{\partial}_* \otimes \tilde{D}_*) - \mathrm{trace}(\tilde{\partial}''_* \otimes \tilde{D}''_*)$$

is a Hochschild boundary. Thus $\mathrm{trace}(\tilde{\partial}'_* \otimes \tilde{D}'_*)$ and $\mathrm{trace}(\tilde{\partial}''_* \otimes \tilde{D}''_*)$ are homologous 1–chains. Let $\delta_* : C_*(\tilde{Y},\tilde{X}) \to C_*(\tilde{Y},\tilde{X})$ be a chain contraction, i.e., $\delta_*\tilde{\partial}''_* + \tilde{\partial}''_*\delta_* = \mathrm{id}_*$.[3] Let $\Delta_* : C_*(\tilde{Y},\tilde{X}) \to C_*(\tilde{Y},\tilde{X})$ be the degree two ϕ-homomorphism[4]:

$$\Delta_* = \delta_*(\delta_*\tilde{f}_* - \tilde{D}''_*).$$

[3] A degree 1 chain homotopy satisfies $D\partial + \partial D = f - g$, as usual. When we put all the matrices together to make a single matrix for ∂ and for D, as in §1(B), our sign rules turn this into $D\phi(\partial) - \partial D = f - g$. Similarly for the degree 2 case with opposite signs. Thus there is no contradiction between the signs in this proof, which are for chain maps rather than matrices, and those in the formula in §1(B) which are for matrices rather than chain maps. Note that the notation $\tilde{D}_*$ has one meaning in §1 and a slightly different meaning here. This should cause no confusion.

[4] A *ϕ–homomorphism* $h : A \to B$ between right G–modules is a homomorphism of the underlying abelian groups such $h(ag) = h(a)\phi(g)$ for all $a \in A$ and $g \in G$.

The equality $\delta_*\tilde{\partial}''_* + \tilde{\partial}''_*\delta_* = \mathrm{id}_*$ implies $\delta_*^2\tilde{\partial}''_* + \delta_*\tilde{\partial}''_*\delta_* = \delta_*$ and $\delta_*\tilde{\partial}''_*\delta_* + \tilde{\partial}''_*\delta_*^2 = \delta_*$ from we which we deduce that $\tilde{\partial}''_*\delta_*^2 = \delta_*^2\tilde{\partial}''_*$. Using this and $\tilde{D}''_*\tilde{\partial}''_* + \tilde{\partial}''_*\tilde{D}''_* = \tilde{f}_* - (\tilde{U}_1)_* = \tilde{f}_*$ (note that $U_1(Y) \subset X$ implies $(\tilde{U}_1)_* = 0$ on $C_*(\tilde{Y}, \tilde{X})$), a calculation yields:

$$\Delta_*\tilde{\partial}''_* - \tilde{\partial}''_*\Delta_* = \tilde{D}''_* - \delta_*\tilde{f}_*,$$

i.e., Δ_* is a degree two ϕ-chain homotopy $\Delta_* : \tilde{D}''_* \simeq \delta_*\tilde{f}_*$. By Lemma 3.2 of [GN$_1$], $\mathrm{trace}(\tilde{\partial}''_* \otimes \delta_*\tilde{f}_*)$ and $\mathrm{trace}(\tilde{\partial}''_* \otimes \tilde{D}''_*)$ are homologous Hochschild 1–chains. Hence

$$\mathcal{E}(f,\tau) = T_1(\tilde{\partial}''_* \otimes \delta_*\tilde{f}_*; G_\phi(f)) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f))$$

which is independent of the choice of H. $\square$

We deduce two corollaries from the proof of Proposition 2.2:

Corollary 2.3. *$\mathcal{E}(f,\tau)$ is represented by the image of* $\mathrm{trace}(\tilde{\partial}''_* \otimes \delta_*\tilde{f}_*)$ *in $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f))$ where $\delta_* : C_*(\tilde{Y}, \tilde{X}) \to C_*(\tilde{Y}, \tilde{X})$ is any chain contraction.* $\square$

Corollary 2.4. *$\mathcal{E}(f,\tau) = R(f \circ H, \tau) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f))$.*

Proof. Retaining the notation of the proof of Proposition 2.2, let $\Delta'_* = \delta_*(\delta_*\tilde{f}_* - \tilde{f}_*\delta_*)$. Then $\Delta'_*\tilde{\partial}''_* - \tilde{\partial}''_*\Delta'_* = \tilde{f}_*\delta_* - \delta_*\tilde{f}_*$ and so, again by Lemma 3.2 of [GN$_1$], $\mathcal{E}(f,\tau)$ is also represented by $\mathrm{trace}(\tilde{\partial}''_* \otimes \tilde{f}_*\delta_*)$. Since $G_\phi(fir) \subset G_\phi(f)$ and $\tilde{\delta}_*$ can be taken to be the chain homotopy associated with the lift of H, this implies that $\mathcal{E}(f,\tau) = R(f \circ H, \tau)$. $\square$

There is a commutative diagram:

(2.5)
$$\begin{array}{ccccc}
\pi_1(X,v) & \xrightarrow{(f|_X)_\#} & \pi_1(X,f(v)) & \xrightarrow{\tau_\#^{-1}} & \pi_1(X,v) \\
\downarrow{\scriptstyle i_\#} & & \downarrow{\scriptstyle i_\#} & & \downarrow{\scriptstyle i_\#} \\
\pi_1(Y,v) & \xrightarrow{f_\#} & \pi_1(Y,f(v)) & \xrightarrow{\tau_\#^{-1}} & \pi_1(Y,v).
\end{array}$$

where the vertical arrows $i_\#$ are isomorphisms. Let $N := \pi_1(X,v)$ and let $\psi : N \to N$ be the composite of horizontal arrows in the first row of (2.5). Then $i_\#\psi = \phi i_\#$ and so $i_\#$ induces a bijection $i_\# : N_\psi \to G_\phi$. It is easy to see that $i_\#(N_\psi(f|_X)) \subset G_\phi(f)$. In this situation we will write $G_\phi(f|_X))$ for $i_\#(N_\psi(f|_X))$ and we will denote by $i_\dagger$ the composite homomorphism

$$\begin{aligned} HH_1(\mathbb{Z}N, (\mathbb{Z}N)^\psi; N_\psi(f|_X)) \quad &\xrightarrow{i_*} \quad HH_1(\mathbb{Z}G,\, (\mathbb{Z}G)^\phi; G_\phi(f|_X))) \\ &\xrightarrow{p} \quad HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f)), \end{aligned}$$

where $p : HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f|_X))) \to HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f))$ is projection as in §1.

The next result formulates the obstruction to homotopy invariance of the one-parameter trace. This result replaces Theorem 4.5 of [GN$_1$] which, as explained in the Introduction, is not valid in the generality asserted there (but see Theorem 2.8 below).

Theorem 2.6. *If the diagram*

$$\begin{tikzcd} X \times I \arrow[r, "F"] \arrow[d, "i\times \mathrm{id}"'] & X \arrow[d, "i"] \\ Y \times I \arrow[r, "E"] & Y \end{tikzcd}$$

is homotopy commutative rel $X \times \{0,1\}$. *Then*

$$R(E,\tau) - i_\dagger R(F,\tau) = p_0\left(\mathcal{E}(E_0,\tau)\right) - p_1\left(\mathcal{E}(E_1,\tau\omega)\right)$$

where τ *is a basepath in* X *from a basepoint* $v \in X$ *to* $F(v,0)$, ω *is the path* $\omega(t) = F(v,t)$, *and, for* $k = 0$ *or* 1, *the homomorphism* $p_k : HH_1(\mathbb{Z}G, (ZG)^\phi; G_\phi(E_k)) \to HH_1(\mathbb{Z}G, (ZG)^\phi; G_\phi(\partial E))$ *is projection.*

Proof. By hypothesis there exists a homotopy $K : (X \times I) \times I \to Y$, $K : iF \simeq E \circ (i \times \mathrm{id})$ rel $X \times \{0,1\}$. Then $r \circ K$ is a homotopy $F \simeq r \circ E \circ (i \times \mathrm{id})$ rel $X \times \{0,1\}$ and so Corollary 4.3 of [GN$_1$] implies that $R(F,\tau) = R(r \circ E \circ (i \times \mathrm{id}),\tau)$. Trivially, there is a commutative diagram:

$$\begin{tikzcd} C_*(\tilde{X}) \arrow[r] \arrow[d, "\tilde{D}_*"'] & C_*(\tilde{Y}) \arrow[r] \arrow[d, "\tilde{D}'_*"'] & C_*(\tilde{Y},\tilde{X}) \arrow[d, "0"'] \\ C_*(\tilde{X}) \arrow[r] & C_*(\tilde{Y}) \arrow[r] & C_*(\tilde{Y},\tilde{X}) \end{tikzcd}$$

where $\tilde{D}$ and $\tilde{D}'$ are the chain homotopies associated to the lifts to the universal cover of $r \circ E \circ (i \times \mathrm{id})$ and $ir \circ E$ respectively. By Proposition 3.5 of [GN$_1$], $\mathrm{trace}(\tilde{\partial}_* \otimes \tilde{D}_*) = \mathrm{trace}(\tilde{\partial}'_* \otimes \tilde{D}'_*)$ where $\tilde{\partial}$ and $\tilde{\partial}'$ are the boundary operators of $C(\tilde{X})$ and $C(\tilde{Y})$ respectively. Since $\mathrm{Fix}(ir \circ E) = \mathrm{Fix}(r \circ E \circ (i \times \mathrm{id}))$,

$$G_\phi(\partial(ir \circ E)) = G_\phi(\partial(r \circ E \circ (i \times \mathrm{id}))) = G_\phi(\partial F)$$

and so $R(F,\tau) = R(ir \circ E,\tau) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(\partial F))$. Define a homotopy $Q : (Y \times I) \times I \to Y$ by $Q = H \circ (E \times \mathrm{id})$. Then $Q_0 = E$ and $Q_1 = ir \circ E$. For $k = 0,1$ define $U^k : Y \times I \to Y$ by $U^k(y,t) = Q(y,k,t)$. Then $U^k = H \circ (E_k \times \mathrm{id})$, $k = 0,1$. By Proposition 4.2 of [GN$_1$], $R(Q_0,\tau) - R(Q_1,\tau) - R(U^0,\tau) + R(U^1,\tau\omega) = 0$ calculated in $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(\partial Q_0 \cup \partial Q_1))$. But $G_\phi(\partial Q_0) = G_\phi(\partial E)$ and $G_\phi(\partial Q_1) = G_\phi(\partial F) \subset G_\phi(\partial E)$. Furthermore, $R(Q_0,\tau) = R(E,\tau)$, $R(Q_1,\tau) = R(ir \circ E,\tau) = i_\dagger R(F,\tau)$, $R(U^0,\tau) = p_0\left(\mathcal{E}(E_0,\tau)\right)$, and $R(U^1,\tau\omega) = p_1\left(\mathcal{E}(E_1,\tau\omega)\right)$. The conclusion of the theorem follows. □

In the special case that $E_0 = E_1 = \mathrm{id}$ the difference $p_0(\mathcal{E}(E_0,\tau)) - p_1(\mathcal{E}(E_1,\tau\omega))$, taking τ to be the constant path at the basepoint v, (we write v for τ), can be expressed in terms of the Dennis trace of the Whitehead torsion of (Y,X). In this case $\phi : G \to G$ is the identity, $G_\phi(E_0) = \{C(1)\}$, and $G_\phi(E_1) = \{C([\omega]^{-1})\}$ where $[\omega]$ is the element of the center of G represented by the closed path $\omega(t) = F(v,t) = E(v,t)$ and, since $\phi = \mathrm{id}$, $C(g)$ is the conjugacy class of $g \in G$. Recall that the Dennis trace induces a homomorphism $\mathrm{DT} : \mathrm{Wh}(G) \to HH_1(\mathbb{Z}G; \{C(1)\})$ (see (1.6)) and that there is an action of the center of G on $HH_1(\mathbb{Z}G)$ (see (1.3)).

Proposition 2.7. *Suppose* $E_0 = E_1 = \mathrm{id}$. *Then*

$$R(E,v) - i_\dagger R(F,v) = p_0(\mathcal{E}(\mathrm{id},v)) - p_1(\mathcal{E}(\mathrm{id},\omega)) = p\big((1 - [\omega])j \circ \mathrm{DT}(\sigma)\big).$$

where $\sigma \in \mathrm{Wh}(G)$ is the Whitehead torsion of the pair (Y, X), and the homormorphisms

$$j : HH_1(\mathbb{Z}G; \{C(1)\}) \to HH_1(\mathbb{Z}G)$$
$$p : HH_1(\mathbb{Z}G) \to HH_1(\mathbb{Z}G; \{C(1), C([\omega]^{-1})\})$$

are, respectively, inclusion and projection. □

A proof of Proposition 2.7 is given in Theorem 2.8 of [GN_2]. (The proof is also implicit in §7 of [GN_1].) Since the torsion of a simple homotopy equivalence vanishes, Proposition 2.7 implies the simple homotopy invariance of the one-parameter trace in an important special case.

Theorem 2.8. *If $E_0 = E_1 = \mathrm{id}$ and the inclusion $i : X \hookrightarrow Y$ is a simple homotopy equivalence then $R(E, v) = i_\dagger R(F, v)$.* □

If $i : X \hookrightarrow Y$ is not a simple homotopy equivalence, the conclusion of Theorem 2.8 can fail to hold:

Example 2.9. Let (p, q) be a pair of relatively prime positive integers with $p > 1$. Recall that the 3–dimensional lens space $L(p, q)$ is the orbit space of the action of the cyclic group $\mathbb{Z}/p = \langle t \mid t^p = 1\rangle$ on the 3–sphere $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$ defined by $t(z_0, z_1) = (e^{2\pi i/p}z_0, e^{2\pi iq/p}z_1)$. Let $[z_0, z_1] \in L(p, q)$ denote the the orbit of $(z_0, z_1) \in S^3$. Define a self homotopy of the identity $\gamma : L(p, q) \times I \to L(p, q)$ by $\gamma([z_0, z_1], s) = [e^{2\pi is/p}z_0, e^{2\pi isq/p}z_1]$. Consider the lens spaces $L(7, 2)$ and $L(7, 1)$ which are homotopy equivalent but not simple homotopy equivalent (see (29.6) and (30.1) of [C]). Let $h : L(7, 1) \to L(7, 2)$ be any homotopy equivalence and let

$$M(h) := (L(7, 1) \times I \overset{\circ}{\to} \cup L(7, 2))/(x, 1) \sim h(x)$$

be the mapping cylinder of h. We identify $L(7, 1)$ with $L(7, 1) \times \{0\} \subset M(h)$. The homotopy $\gamma : L(7, 1) \times I \to L(7, 1)$ can be extended to a self homotopy of the identity, $\gamma' : M(h) \times I \to M(h)$. Choose a basepoint $v \in L(7, 1)$. By Remark 5.8 of [GN_1], the difference $R(\gamma', v) - i_\dagger R(\gamma, v)$ is not zero.

Recall the homomorphism $\mathcal{D} : H_1(GL(\mathbb{Z}G), M(\mathbb{Z}G)^\phi) \to HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ from (1.7).

Proposition 2.10. *Suppose $f : (Y, X) \to (Y, X)$ is such that there exists a strong deformation retraction H with the additional property that $f \circ H = H \circ (f \times \mathrm{id})$. Then there is an $\alpha \in H_1(GL(\mathbb{Z}G), M(\mathbb{Z}G)^\phi)$ such that $\mathcal{E}(f, \tau) = p \circ \mathcal{D}(\alpha)$ where $p : HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \to HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f))$ is projection.*

Proof. As before, let $(C_*(\tilde{Y}, \tilde{X}), \tilde{\partial}_*)$ be the cellular chain complex of the pair $(\tilde{Y}, \tilde{X})$. Let $\tilde{f}_*$ be the chain map associated to the lift of f and let $\tilde{\delta}_*$ be the chain homotopy associated to the lift of H. Note that $\tilde{\delta}_*$ is a chain contraction. The hypothesis $f \circ H = H \circ (f \times \mathrm{id})$ implies that $\tilde{f}_*\tilde{\delta}_* = \tilde{\delta}_*\tilde{f}_*$. Replacing $\tilde{\delta}_*$ with the chain contraction $\tilde{\delta}_*\tilde{\partial}_*\tilde{\delta}_*$, we can assume that $\tilde{\delta}_*^2 = 0$ and the property $\tilde{f}_*\tilde{\delta}_* = \tilde{\delta}_*\tilde{f}_*$ is retained. Let B and F be the $\mathbb{Z}G$–matrices:

$$B = \begin{bmatrix} \tilde{\partial}_1 & 0 & 0\cdots \\ \tilde{\delta}_1 & \tilde{\partial}_3 & 0\cdots \\ 0 & \tilde{\delta}_3 & \ddots \\ \vdots & 0 & \ddots \end{bmatrix}, \qquad F = \begin{bmatrix} \tilde{f}_0 & 0 & 0\cdots \\ 0 & \tilde{f}_2 & 0\cdots \\ 0 & 0 & \tilde{f}_4\cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Then B is invertible and by Proposition 3.7 of [GN$_1$], $\mathrm{trace}(\tilde{\partial}_* \otimes \tilde{\delta}_* \tilde{f}_*)$ is homologous to $-\,\mathrm{trace}(B \otimes B^{-1}F)$. View $-(B - \mathrm{id}) \otimes F$ as a 1–cycle representing an element α of $H_1(GL(\mathbb{Z}G), M(\mathbb{Z}G)^{\phi})$. Then $\mathcal{D}(\alpha)$ is the Hochschild homology class of $-\,\mathrm{trace}(B \otimes B^{-1}F)$. Since $\mathcal{E}(f,\tau)$ is represented by $\mathrm{trace}(\tilde{\partial}_* \otimes \tilde{\delta}_* \tilde{f}_*)$, the conclusion of the proposition follows. $\square$

Remark. The point of Proposition 2.10 is that the group $H_1(GL(\mathbb{Z}G), M(\mathbb{Z}G)^{\phi})$ is analogous to $K_1(\mathbb{Z}G) = H_1(GL(\mathbb{Z}G), \mathbb{Z})$ and the element $\alpha \in H_1(GL(\mathbb{Z}G), M(\mathbb{Z}G)^{\phi})$ constructed in its proof is reminiscent of Whitehead torsion.

§3. Failure of Simple Homotopy Invariance

In this section we describe a finite CW pair (Y, X) and homotopies $F : X \times I \to X$, $E : Y \times I \to Y$ so that the diagram

$$\begin{tikzcd} X \times I \arrow[r, "F"] \arrow[d, "i\times\mathrm{id}"'] & X \arrow[d, "i"] \\ Y \times I \arrow[r, "E"] & Y \end{tikzcd}$$

is homotopy commutative rel $X \times \{0,1\}$ and $i : X \hookrightarrow Y$ is a simple homotopy equivalence, but where $R(E,\tau) \neq i_{\dagger} R(F,\tau)$. In fact Y will be obtained from X by two elementary expansions (see [C]). Lemma 4.7 of [GN$_1$] implies that if Y were obtained from X by a single elementary expansion then $R(E,\tau) = i_{\dagger} R(F,\tau)$, so the example given here would appear to be as economical as possible.

Let $X = P^2$. The space Y is $(P^2 \vee B^3 \vee S^2) \cup_h e^3$ where P^2 is the projective plane, S^2 is the 2–sphere, B^3 and e^3 are 3–cells and h is a certain attaching map for e^3. Give X the following cell structure: one vertex v, one 1–cell attached to v to give S^1, and one 2–cell attached to S^1 via a map of degree two yielding $P^2 = X$. We now specify the cell structure of Y. Attach two 2–cells to X via constant maps at v to give $P^2 \vee S^2_1 \vee S^2_2$, attach one 3–cell e^3_1 by a homeomorphism $S^2 \to S^2_1$ to give $P^2 \vee B^3 \vee S^2_2$, and attach one 3–cell e^3_2 by a map h into the 2–skeleton; this map will be specified shortly.

In this case $G := \pi_1(Y, v) = \langle \omega \mid \omega^2 = 1 \rangle$, the group of order 2 and $\phi = \mathrm{id} : G \to G$. Choose orientations for the cells of Y and a lift of each cell to the universal cover $\tilde{Y}$. Orient $\tilde{e}$ (the lift of e) compatibly with e, and $\tilde{e}\omega$ so that the covering transformation ω preserves orientation. The basepoint of $\tilde{Y}$ is chosen to be $\tilde{v}$. The first requirement for h is that on cellular chains,

$$\tilde{\partial}\tilde{e}^3_2 = \tilde{S}^2_1(1-\omega) + \tilde{S}^2_2.$$

Let $Z = P^2 \vee B^3$, a subcomplex of Y. The second requirement for h is that the inclusion $i_2 : Z \hookrightarrow Y$ be an elementary expansion; this is possible because the coefficient of $\tilde{S}^2_2$ in $\tilde{\partial}\tilde{e}^3_2$ is 1. Of course, $i_1 : X \hookrightarrow Z$ is also an elementary expansion. Let $Y @ > r_2 >> Z @ > r_1 >> X$ be collapsing retractions. The matrix for $\tilde{\partial}_3 : C_3(\tilde{Y}, \tilde{X}) \to C_2(\tilde{Y}, \tilde{X})$ is

$$\begin{bmatrix} 1 & 1-\omega \\ 0 & 1 \end{bmatrix}.$$

Define $f : (Y, X) \to (Y, X)$ to be a cellular map which restricts to id_X on X and induces the matrix

$$\begin{bmatrix} 0 & 0 \\ 1+\omega & 0 \end{bmatrix}.$$

both on $C_2(\tilde{Y}, \tilde{X})$ and on $C_3(\tilde{Y}, \tilde{X})$. Such a map f exists because the two displayed matrices commute. Furthermore, we may choose f to have no fixed points outside X because the diagonal terms in the latter matrix are 0. We have $r_1 r_2 f i_2 i_1 \simeq \mathrm{id}_X$ rel X, and $i_2 i_1 r_1 r_2 \simeq \mathrm{id}_Y$ rel X, so $f \simeq \mathrm{id}_Y$ rel X. Let $E : Y \times I \to Y$ be a homotopy rel X with $E_0 = f$ and $E_1 = \mathrm{id}_Y$. We wish to apply Theorem 2.6 to this E, where $F : X \times I \to X$ is projection. In this case it is clear that the diagram in the statement of Theorem 2.6 is strictly commutative. Moreover, we may take τ and ω to the constant paths at the basepoint v (denoting them by v). With notation as in Theorem 2.6 we have:

Proposition 3.1. *$R(E, v) - i_\dagger R(F, v) = p_0(\mathcal{E}(f, v)) \in HH_1(\mathbb{Z}G; G_{\mathrm{id}}(\partial E))$, where $i = i_2 i_1 : X \hookrightarrow Y$.*

Remark. When, as in this case, $\phi = \mathrm{id}$, we write $HH_1(\mathbb{Z}G; G_{\mathrm{id}}(\partial E))$ for $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(\partial E))$.

Proof. We apply Theorem 2.6. The term $p_1(\mathcal{E}(\mathrm{id}_Y, v)) = 0$ because H, the concatenation of two elementary collapses, can be chosen to have no fixed points associated with semiconjugacy classes outside $G_{\mathrm{id}}(\partial H)$. □

It remains to prove that the term $p_0(\mathcal{E}(f, v))$ in Proposition 3.1 is not zero. For this, we return to the general situation of §2. So, until after Proposition 3.3, X and Y are general complexes as in that section. We consider special cases.

Special Case 1. $f(X) \subset X$ (as in §2) and $i : X \hookrightarrow Y$ is an elementary expansion. Let $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ index the $(q-1)$–cells and q–cells of Y respectively. Assume $Y = X \cup e^{q-1}_{\alpha_0} \cup e^q_{\beta_0}$ where $e^{q-1}_{\alpha_0}$ is the free face of $e^q_{\beta_0}$ (see p.14 of [C]). Then $C_q(\tilde{Y}, \tilde{X})$ is a free $\mathbb{Z}G$–module of rank 1 in dimensions $q-1$ and q and is zero in all other dimensions. Moreover, we may choose lifts and orientations so that $\tilde{e}^{q-1}_{\alpha_0} \subset \tilde{e}^q_{\beta_0}$. Then the unique non-zero entry in the matrix of δ_*, the chain contraction of $C_*(\tilde{Y}, \tilde{X})$ induced by the collapse Y onto X, is $(\delta_*)_{\beta_0\alpha_0} = \pm 1$ and

$$(\delta_* \tilde{f}_*)_{\beta\alpha} = \sum_\nu (\delta_*)_{\beta\nu} (\tilde{f}_*)_{\nu\alpha} = \begin{cases} \pm(\tilde{f}_*)_{\alpha_0\alpha} & \text{if } \beta = \beta_0 \\ 0 & \text{if } \beta \neq \beta_0 \end{cases}$$

The signs depend on choices and will be irrelevant. The unique non-zero entry in the matrix of $\tilde{\partial}''_*$, the boundary operator of $C_*(\tilde{Y}, \tilde{X})$, is $(\tilde{\partial}''_*)_{\alpha_0\beta_0} = \pm 1$. Thus we have:

Proposition 3.2. *In this situation,* $\mathrm{trace}(\tilde{\partial}''_* \otimes \delta_* \tilde{f}_*) = \pm 1 \otimes (\tilde{f}_*)_{\alpha_0\alpha_0}$ *which is a Hochschild boundary.* □

Special Case 2. i is an elementary expansion but it is not assumed that $f(X) \subset X$. There is a homotopy $U := H \circ (f \times \mathrm{id}) : Y \times I \to Y$ as before which might not be constant on $X \times I$. As in the proof of Proposition 2.2, we denote by $\tilde{D}'_* : C_*(\tilde{Y}) \to C_*(\tilde{Y})$ the chain homotopy associated with U, and $\tilde{\partial}'_*$ the boundary operator of $C_*(\tilde{Y})$.

Proposition 3.3. *Under these hypotheses, the Hochschild 1–chains* $\operatorname{trace}(\tilde\partial'_*\otimes\tilde D'_*)$ *and* $\sum_{\alpha\in\mathcal{A}}\left(\pm(\tilde\partial'_*)_{\alpha\beta_0}\otimes(\tilde f_*)_{\alpha_0\alpha}\right)$ *are homologous for suitable choices of sign in each term of the sum.*

Proof. The proof runs parallel to that of Proposition 2.2 and we stay as close to it as possible. Now we must work with $C_*(\tilde Y)$ rather than with $C_*(\tilde Y,\tilde X)$ and so

$$\delta_*\tilde\partial'_*+\tilde\partial'_*\delta_*=\mathrm{id}_*-\tilde r_*.$$

Let $\Delta'_*: C_*(\tilde Y)\to C_*(\tilde Y)$ be the degree two ϕ–homomorphism $\Delta'_*=\delta_*(\delta_*\tilde f_*-\tilde D'_*)$. Using $\tilde D'_*\tilde\partial'_*+\tilde\partial'_*\tilde D'_*=\tilde f_*-\tilde r_*\tilde f_*$, a calculation yields

$$\Delta'_*\tilde\partial'_*-\tilde\partial'_*\Delta'_*=(\mathrm{id}_*-\tilde r_*)(\tilde D'_*-\delta_*\tilde f_*),$$

i.e., Δ'_* is a degree two ϕ–chain homotopy $\Delta'_*:\tilde D'_*\simeq\delta_*\tilde f_*-\tilde r_*(\delta_*\tilde f_*-\tilde D'_*)$. We may assume that the retraction r takes the q-chain $\tilde e^q_{\beta_0}$ to 0, and hence that $\tilde r_*(\delta_*\tilde f_*-\tilde D'_*)=0$. Thus $\Delta'_*:\tilde D'_*\simeq\delta_*\tilde f_*$. By Lemma 3.2 of [GN$_1$], $\operatorname{trace}(\tilde\partial'_*\otimes\delta_*\tilde f_*)$ and $\operatorname{trace}(\tilde\partial'_*\otimes\tilde D'_*)$ are homologous Hochschild 1–chains. We now proceed as in the proof of Proposition 3.2 (noting that the matrix of $\tilde\partial'_*$ can have many non-zero entries) to obtain

$$\operatorname{trace}(\tilde\partial'_*\otimes\delta_*\tilde f_*)=\sum_{\alpha\in\mathcal{A}}\left(\pm(\tilde\partial'_*)_{\alpha\beta_0}\otimes(\tilde f_*)_{\alpha_0\alpha}\right). \qquad\square$$

Remark 3.4. If the chain contractions δ_* are chosen to be induced by the elementary collapses H in Proposition 3.3, then $\delta\tilde f_*=\tilde D'_*$. This simplifies the above proof since the degree two chain homotopies Δ'_* become zero. However, one must then prove that the homotopy $H(f\times\mathrm{id})$ induces the chain homotopy $\delta_*\tilde f$, something which is true but not entirely obvious.

We now return to the particular map $f:(Y,X)\to(Y,X)$ constructed earlier in this section. Let $H^{(2)}:Y\times I\to Y$ and $H^{(1)}:Z\times I\to Z$ be strong deformation retractions corresponding to the collapses r_2 and r_1; i.e., $H^{(2)}_0=\mathrm{id}_Y$, $H^{(2)}_1=i_2r_2$, $H^{(1)}_0=\mathrm{id}_Z$, and $H^{(1)}_1=i_1r_1$. Let $H:Y\times I\to Y$ be the homotopy obtained by concatenating $H^{(2)}$ and $i_2\circ H^{(1)}\circ(r_2\times\mathrm{id})$:

$$H(y,t)=\begin{cases}H^{(2)}(y,2t) & \text{if } 0\le t\le\frac12\\ i_2\circ H^{(1)}(r_2(y),2t-1) & \text{if } \frac12\le t\le 1.\end{cases}$$

Note that $H_0=\mathrm{id}_Y$ and $H_1=i_2i_1r_1r_2$. Writing $U:=H\circ(f\times\mathrm{id})$, we compute $\mathcal{E}(f,v):=R(U,v)$ by adding traces whose form is given by Propositions 3.2 and 3.3 corresponding to the two parts of the homotopy. The contribution from $i_2\circ H^{(1)}\circ(r_2\times\mathrm{id})$ can be ignored as it is a Hochschild boundary. There are exactly two conjugacy classes $C(1)$ and $C(\omega)$ in $G\cong\mathbb{Z}/2\mathbb{Z}$. Since $f|_X=\mathrm{id}_X$ and f has no other fixed points, $G_{\mathrm{id}}(f)=\{C(1)\}$. Thus in computing $\mathcal{E}(f,v)$ we may ignore all Hochschild homology classes whose marker is $1\in G$. It follows that in the sum given in Proposition 3.3, only the term $\pm\omega\otimes1=\pm\omega\otimes\omega\omega$ matters; its marker is $\omega\in G$. By Proposition 1.2, this term is non-trivial in $HH_1(\mathbb{Z}G;G_1(f))$. Thus we have:

Theorem 3.5. *In this example,* $R(E,v)\neq i_\dagger R(F,v)$*.* $\square$

Since the inclusion $i : X \hookrightarrow Y$ is a simple homotopy equivalence (having been obtained by two elementary expansions), Theorem 3.4 provides an example of the failure of simple homotopy invariance of the 1–parameter trace.

§4. Proof of Combinatorial Invariance

In this section we prove the combinatorial invariance of $R(F)$, Theorem 4.6.

Let $X = Z_0 \subset Z_1 \cdots \subset Z_N = Y$ be a filtration of Y by subcomplexes such that Z_{i-1} is a strong deformation retract of Z_i for $i = 1, \ldots, N$. Let $v \in X$ be a basepoint. We identify $\pi_1(Z_i, v)$ with $G := \pi_1(Y, v)$ via the isomorphism induced by the inclusion $Z_i \subset Y$. Let $f : (Y, X) \to (Y, X)$ be a filtration preserving cellular map, i.e., $f(Z_i) \subset Z_i$ for $i = 0, \ldots, N$. Let $f_i : (Z_i, Z_{i-1}) \to (Z_i, Z_{i-1})$, $i = 1, \ldots, N$, denote the restriction of f to Z_i regarded as map of pairs. Let τ be a basepath in X from v to $f(v)$. Since $f_i = f|_{Z_i}$, it follows that $G_\phi(f_i) \subset G_\phi(f)$ for $i = 1, \ldots, N$ (see the discussion following (2.5)). Hence there are projections $p_i : HH_1(\mathbb{Z}G, \mathbb{Z}G^\phi; G_\phi(f_i)) \to HH_1(\mathbb{Z}G, \mathbb{Z}G^\phi; G_\phi(f))$ for $i = 1, \ldots, N$.

Proposition 4.1. *For $f : (Y, X) \to (Y, X)$ as above, $\mathcal{E}(f, \tau) = \sum_{i=1}^{N} p_i(\mathcal{E}(f_i, \tau))$.*

Proof. For $i = 1, \ldots, N$, choose strong deformation retractions $H_i : Z_i \times I \to Z_i$ of Z_i into Z_{i-1} and strong deformation retractions $Q_i : Y \times I \to Y$ of Y into Z_i. The homotopy $F_i : Y \times I \to Y$ defined by

$$F_i(y,t) = \begin{cases} Q_i(y, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ H_i(Q_i(y,1), 2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

is a strong deformation retraction of Y into Z_{i-1}. Note that $F_i(Z_i \times I) \subset Z_i$. Let $g_i = f$, regarded as a map of pairs $g_i : (Y, Z_i) \to (Y, Z_i)$. Let $\tilde{Y}$ be the universal cover of Y and let $\tilde{Z}_i \subset \tilde{Y}$ be the universal cover of Z_i. Consider the diagram

$$\begin{tikzcd} C_*(\tilde{Z}_i, \tilde{Z}_{i-1}) \arrow[r] \arrow[d, "\tilde{D}^i_*"'] & C_*(\tilde{Y}, \tilde{Z}_{i-1}) \arrow[r] \arrow[d, "\tilde{P}^i_*"'] & C_*(\tilde{Y}, \tilde{Z}_i) \arrow[d, "\tilde{T}^i_*"'] \\ C_*(\tilde{Z}_i, \tilde{Z}_{i-1}) \arrow[r] & C_*(\tilde{Y}, \tilde{Z}_{i-1}) \arrow[r] & C_*(\tilde{Y}, \tilde{Z}_i) \end{tikzcd}$$

where $\tilde{D}^i_*$, $\tilde{P}^i_*$, and$\tilde{T}^i_*$ are the chain homotopies associated with the lifts of $H_i \circ (f_i \times \mathrm{id})$, $F_i \circ (g_{i-1} \times \mathrm{id})$ and $F_i \circ (g_i \times \mathrm{id})$ respectively. Since $H_i \circ (f_i \times \mathrm{id}) \simeq F_i \circ (g_{i-1} \times \mathrm{id})|_{(Z_i, Z_{i-1}) \times I}$ rel $Z_i \times \{0,1\} \cup Z_{i-1} \times I$, the left hand square of the above diagram is chain homotopy commutative. The right hand square is strictly commutative. By Proposition 3.5 of [GN$_1$] (which applies by Addendum 3.6 of [GN$_1$] since $C_*(\tilde{Y}, \tilde{Z}_i)$ is a contractible chain complex),

$$\mathrm{trace}(\tilde{\partial}^{i-1}_* \otimes \tilde{P}^i_*) - \mathrm{trace}(\tilde{\partial}^i_* \otimes \tilde{T}^i_*) - \mathrm{trace}(\tilde{\partial}'^i_* \otimes \tilde{D}^i_*)$$

is a Hochschild boundary where $\tilde{\partial}^k_*$, for $k = i-1$ or i, is the boundary operator of $C_*(\tilde{Y}, \tilde{Z}_k)$ and $\tilde{\partial}'^i_*$ is the boundary operator of $C_*(\tilde{Z}_i, \tilde{Z}_{i-1})$. Although F_i is not necessarily a strong deformation of Y into Z_i, $\mathcal{E}(g_i, \tau)$ can be computed from $\mathrm{trace}(\tilde{\partial}^i_* \otimes \tilde{T}^i_*)$ because the chain homotopy $(\tilde{F}_i)_* : C_*(\tilde{Y}, \tilde{Z}_i) \to C_*(\tilde{Y}, \tilde{Z}_i)$ induced by the lift of F_i is a chain contraction and so Corollary 2.3 applies. Hence

$$\mathcal{E}(g_{i-1}, \tau) = \mathcal{E}(g_i, \tau) + p_i(\mathcal{E}(f_i, \tau)) \ \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(f)). \tag{4.2}$$

Summing (4.2) yields

$$\sum_{i=1}^{N} \mathcal{E}(g_{i-1}, \tau) = \sum_{i=1}^{N} \mathcal{E}(g_i, \tau) + \sum_{i=1}^{N} p_i(\mathcal{E}(f_i, \tau))$$

and so $\mathcal{E}(g_0, \tau) - \mathcal{E}(g_N, \tau) = \sum_{i=1}^{N} p_i(\mathcal{E}(f_i, \tau))$. Clearly $\mathcal{E}(g_N, \tau) = 0$. Since $f = g_0$ the conclusion of the Proposition follows. $\square$

Definition 4.3. A *subdivision* X' of a CW complex X is another CW structure on the underlying space of X such that each open cell of X' is contained in an open cell of X.

Suppose that X is a finite connected CW complex, X' is a subdivision of X, and $f : X \to X$ is a self-map of the underlying space which is cellular for both CW structures, i.e., $f : X \to X$ and $f : X' \to X'$ are both cellular. Let Y be the CW complex whose underlying space is $X \times I$ and whose open j-cells are of the form $\overset{\circ}{\rightarrow} e^j \times \{0\}$, $\overset{\circ}{\rightarrow} e^{j-1} \times \overset{\circ}{\rightarrow} I$, or $\overset{\circ}{\rightarrow} s^j \times \{1\}$ where $\overset{\circ}{\rightarrow} e^k$, for $k = j-1$ or j, is an open k-cell of X and $\overset{\circ}{\rightarrow} s^j$ is an open j-cell of X'. There is a natural identification of Y with the mapping cylinder of the identity map $X \to X'$ (which is cellular); informally, the CW structure of Y is obtained from the product CW structure of $X \times I$ by subdividing $X \times \{1\}$ according to X'. Since f is cellular with respect to both CW structures, $f \times \mathrm{id} : Y \to Y$ is cellular. Let $\hat{f} : (Y, X \times \{0\}) \to (Y, X \times \{0\})$ and $\hat{f}' : (Y, X' \times \{1\}) \to (Y, X' \times \{1\})$ be $f \times \mathrm{id}$ regarded as maps of pairs. Let the vertex $v \in X$ be a basepoint and τ a basepath in X from v to $f(v)$. Choose $(v, 0)$ as the basepoint for $X \times \{0\}$ and $(v, 1)$ as the basepoint for $X' \times \{1\}$. Define basepaths $\hat{\tau}$ and $\hat{\tau}'$ by $\hat{\tau}(t) = (\tau(t), 0)$ and $\hat{\tau}'(t) = (\tau(t), 1)$, $t \in [0, 1]$.

Proposition 4.4. *For $\hat{f}$ and $\hat{f}'$ as above, $\mathcal{E}(\hat{f}, \hat{\tau}) = 0$ and $\mathcal{E}(\hat{f}', \hat{\tau}') = 0$.*

Proof. Define filtrations $Z_0 \subset Z_1 \cdots \subset Z_N = Y$ and $Z'_0 \subset Z'_1 \cdots \subset Z'_N = Y$ of Y by $Z_j = X \times \{0\} \cup X^{j-1} \times I$ and $Z'_j = X' \times \{1\} \cup X^{j-1} \times I$, $j = 0, \ldots, N$, where X^j is the j–skeleton of X (by convention, $X^{-1} := \emptyset$) and $N = \dim Y = \dim X + 1$. Note that $\hat{f}(Z_j) \subset Z_j$ and $\hat{f}'(Z'_j) \subset Z'_j$, $j = 0, \ldots, N$. Clearly, $Z_{j-1} \subset Z_j$ and $Z'_{j-1} \subset Z'_j$ are strong deformation retracts. By Proposition 4.1,

$$\mathcal{E}(\hat{f}, \hat{\tau}) = \sum_{i=1}^{N} p_i(\mathcal{E}(\hat{f}_i, \hat{\tau})) \quad \text{and} \quad \mathcal{E}(\hat{f}', \hat{\tau}') = \sum_{i=1}^{N} p_i(\mathcal{E}(\hat{f}'_i, \hat{\tau}')) \tag{4.5}$$

where $\hat{f}_i : (Z_i, Z_{i-1}) \to (Z_i, Z_{i-1})$ and $\hat{f}'_i : (Z'_i, Z'_{i-1}) \to (Z'_i, Z'_{i-1})$ are, respectively, the restriction of $\hat{f}$ to Z_i and of $\hat{f}'$ to Z'_i, regarded as a maps of pairs. We will show that $\mathcal{E}(\hat{f}_i, \hat{\tau}) = 0$ and $\mathcal{E}(\hat{f}'_i, \hat{\tau}') = 0$ from which the conclusion of the Proposition then follows from (4.5). As in the proof of Proposition 4.1, let $\tilde{Z}_i \subset \tilde{Y}$ be the universal cover of Z_i. Lifts of the cells of Y can be chosen so that the matrix of the boundary operator $\tilde{\partial}^i_* : C_*(\tilde{Z}_i, \tilde{Z}_{i-1}) \to C_*(\tilde{Z}_i, \tilde{Z}_{i-1})$ is an integer matrix (i.e., its entries belong to $\mathbb{Z} \hookrightarrow \mathbb{Z}G$) as follows. Choose lifts of the cells of $X \times \{0\}$. These lifts determine lifts of the cells in the product CW complex $X \times I$ in the obvious manner. Since Y is a subdivision of $X \times I$, this determines lifts for all the cells of Y. The integrality of the matrix of $\tilde{\partial}^i_*$ implies that $\mathcal{E}(\hat{f}_i, \hat{\tau}) = 0$ because by Corollary 2.3 $\mathcal{E}(\hat{f}_i, \hat{\tau})$ is represented by a Hochschild 1–chain of the form $\mathrm{trace}(\tilde{\partial}^i_* \otimes \tilde{E}_*)$ which is a sum of chains of the form $1 \otimes u$, $u \in \mathbb{Z}G$; this a Hochschild boundary by Lemma 1.1. Similarly, $\mathcal{E}(\hat{f}'_i, \hat{\tau}') = 0$. $\square$

Corollary 4.6 of [GN$_1$] is correct as stated there, but it requires a new proof:

Theorem 4.6 (**Combinatorial invariance**). *Suppose $F : X \times I \to X$ is a cellular homotopy and X' is a subdivision of X such that $F_i : X' \to X'$ is cellular for $i = 0, 1$. If $F' : X' \times I \to X'$ is a cellular homotopy which is homotopic to F rel $X' \times \{0,1\}$ then $R(F, \tau) = R(F', \tau)$ where τ is a basepath in X from v to $F_0(v)$.*

Proof. Let $H : (X \times I) \times I \to X$ be a homotopy $H : F \simeq F'$ rel $X \times \{0,1\}$. Then for $x \in X$ and $s, t \in I$, $H(x,s,0) = F(x,s)$, $H(x,s,1) = F'(x,s)$, $H(x,0,t) = F_0(x) = F'_0(x)$, and $H(x,1,t) = F_1(x) = F'_1(x)$. As in Proposition 4.4, let Y be the CW complex whose underlying space is $X \times I$ and whose CW structure is obtained from the product CW structure of $X \times I$ by subdividing $X \times \{1\}$ according to X'. Define $\bar{E} : Y \times I \to Y$ by $\bar{E}((x,t),s) = (H(x,s,t),t)$ for $x \in X$ and $s, t \in I$. Note that $\bar{E}$ is cellular on the subcomplex $A := (X \times \{0\}) \times I \,\cup\, (X' \times \{1\}) \times I \,\cup\, Y \times \{0,1\}$ of $Y \times I$ and so by the Cellular Approximation Theorem, $\bar{E}$ is homotopic rel A to a cellular homotopy $E : Y \times I \to Y$. There are (strictly) commutative diagrams:

$$\begin{tikzcd}
(X \times \{0\}) \times I \arrow[r, "F"] \arrow[d, "i \times \mathrm{id}"'] & X \times \{0\} \arrow[d, "i"] \\
Y \times I \arrow[r, "E"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(X' \times \{1\}) \times I \arrow[r, "E"] \arrow[d, "j \times \mathrm{id}"'] & X' \times \{1\} \arrow[d, "j"] \\
Y \times I \arrow[r, "E"] & Y
\end{tikzcd}$$

where, with a slight abuse of notation, $F : (X \times \{0\}) \times I \to X \times \{0\}$ is the map given by $F((x,0),t) = (F(x,t),0)$ and $F' : (X' \times \{1\}) \times I \to X' \times \{1\}$ is the map given by $F'((x,1),t) = (F'(x,t),1)$; the vertical maps are inclusions of subcomplexes. Note that $E_k = F_k \times \mathrm{id} = F'_k \times \mathrm{id}$ for $k = 0,1$ and so $\mathrm{Fix}(E_k) = \mathrm{Fix}(F_k) \times I$ for $k = 0,1$. Define paths $\hat{\tau}_0$ and $\hat{\tau}_1$ in $X \times \{0\}$ and paths $\hat{\tau}'_0$ and $\hat{\tau}'_1$ in $X' \times \{1\}$ by $\hat{\tau}_0 = (\tau, 0)$, $\hat{\tau}_1 = (\tau\omega, 0)$, $\hat{\tau}'_0 = (\tau, 1)$, and $\hat{\tau}'_1 = (\tau\omega', 1)$ where $\omega(t) = F(v,t)$ and $\omega'(t) = F'(v,t)$. As in the discussion preceding Proposition 4.4, for $k = 0,1$ let $\hat{F}_k : (Y, X \times \{0\}) \to (Y, X \times \{0\})$ and $\hat{F}'_k : (Y, X' \times \{1\}) \to (Y, X' \times \{1\})$ denote $F_k \times \mathrm{id}$, regarded as maps of pairs. Let $G := \pi_1(Y, (v,0))$, $G' := \pi_1(Y, (v,1))$, $\phi : G \to G$ the homomorphism induced by E and $\hat{\tau}_0$, and $\phi' : G' \to G'$ the homomorphism induced by E and $\hat{\tau}'_0$. By Theorem 2.6,

$$\begin{aligned}
R(E, \hat{\tau}_0) - i_\dagger R(F, \tau) &= p_0(\mathcal{E}(\hat{F}_0, \hat{\tau}_0)) - p_1(\mathcal{E}(\hat{F}_1, \hat{\tau}_1)) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(\partial E)) \\
R(E, \hat{\tau}'_0) - j_\dagger R(F', \tau) &= p_0(\mathcal{E}(\hat{F}'_0, \hat{\tau}'_0)) - p_1(\mathcal{E}(\hat{F}'_1, \hat{\tau}'_1)) \in HH_1(\mathbb{Z}G', (\mathbb{Z}G')^{\phi'}; G'_{\phi'}(\partial E))
\end{aligned}$$

By Proposition 4.4, the right sides of these equalities vanish and so $R(E, \hat{\tau}_0) = i_\dagger R(F, \tau)$ and $R(E, \hat{\tau}'_0) = j_\dagger R(F', \tau)$. Let $\sigma : I \to Y$ be the path $\sigma(t) = (v, t)$. Then σ induces an isomorphism $\sigma_\# : G \to G'$ such that $\phi'\sigma_\# = \sigma_\#\phi$ which in turn induces an isomorphism $\sigma_* : HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; G_\phi(\partial E)) \to HH_1(\mathbb{Z}G', (\mathbb{Z}G')^{\phi'}; G'_{\phi'}(\partial E))$ (see §1). Furthermore, $\sigma_* R(E, \hat{\tau}_0) = R(E, \hat{\tau}'_0)$ and hence $\sigma_* i_\dagger R(F, \tau) = j_\dagger R(F', \tau)$. Now $\sigma_* i_\dagger = j_\dagger$ and $j_\dagger$ is an isomorphism since $G_\phi(\partial F) = G_\phi(\partial F') = G_\phi(E)$ and so it follows that $R(F, \tau) = R(F', \tau)$. $\square$

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